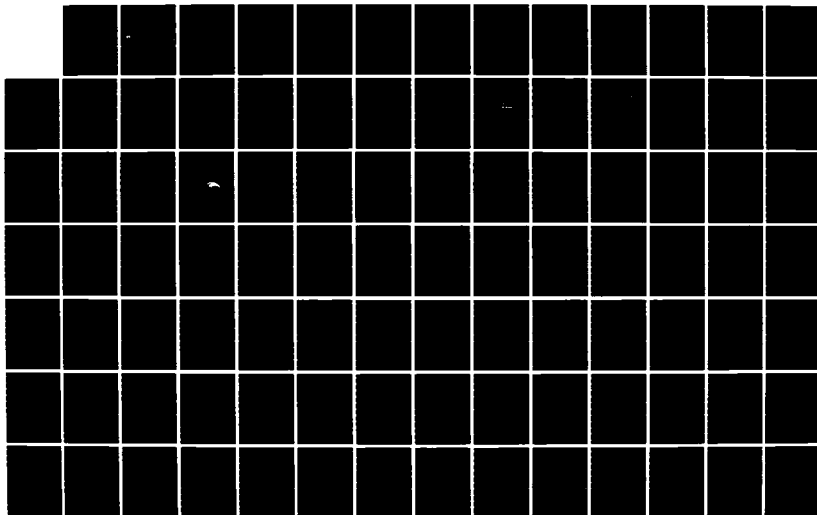AD-A167 630   MODELING AND ANALYSIS OF UNCERTAINTIES IN SURVIVABILITY   1/2
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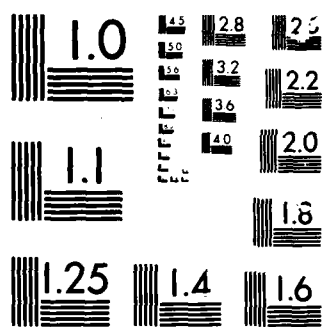   PALO ALTO CA   F S WONG MAR 86 R-8503 AFWL-TR-85-84
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MODELING AND ANALYSIS OF UNCERTAINTIES IN SURVIVABILITY AND VULNERABILITY ASSESSMENT

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March 1986

Final Report

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AIR FORCE WEAPONS LABORATORY
Air Force Systems Command
Kirtland Air Force Base, NM   87117-6008

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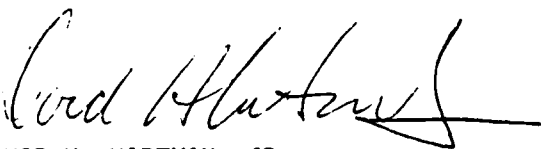
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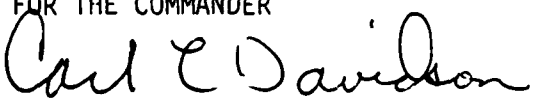
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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This study is based on the belief that not all uncertainties encountered in survivability and vulnerability assessment of protective structures are random. It is argued that at least two types of uncertainties should be addressed: random and nonrandom. Characteristics of random uncertainties include a well-defined event and sample space, repeatable experiments under identical conditions and meaningful sample statistics. Nonrandom uncertainties have different characteristics: the sample space is not well-defined, there are few samples to evaluate, and sample mean and variance do not have physical meaning. Examples of nonrandom uncertainties are vague and imprecise information, linguistic data, subjective judgment, and extreme complexity or details. Uncertainties associated with most analysis models are nonrandom.

Accordingly, two extensions to current statistical assessment methodologies are pursued in the study: to improve random models of uncertainties by using stocastic methods; (over)

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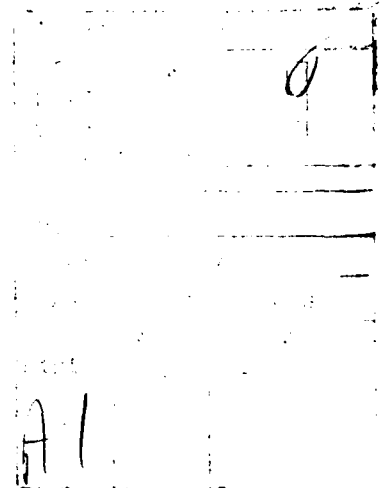
19. ABSTRACT (Continued)

and to expand current methodologies by incorporating nonrandom uncertainties. The first extension leads to a study of random equations and their solution techniques. Emphasis of the work is on numerical stochastic analysis and, in particular, the stochastic difference equations and other discretization methods. The second extension leads to a study of fuzzy sets as an adequate and natural approach to modeling nonrandom uncertainties. The basic concept of fuzzy set theory is described. Suggestions for use of fuzzy models of uncertainties in survivability assessment and how they can be integrated with random models to form a more complete (and, it is argued, better) assessment approach are given.

The use of numerical stochastics in assessment is promising. However, because it is extremely difficult mathematical problem, the numerical methods developed are not yet mature enough for operational applications. Much more fundamental research needs to be done. Fuzzy set modeling, on the other hand, lends itself readily to practical engineering application, as shown by the two examples described in the report. In the first project, modeling uncertainties (how well analytic models compare with the real world) are treated using fuzzy descriptions of their gravity and effect on the computed response. All expressed opinions on different features of the model are summarized in the form of fuzzy relations, which then embody the modeling uncertainties. The second project demonstrates an approach to solicit and aggregate expert opinions in assessing damage to structures. The implementation of these assessments and other available data (deterministic, random and others) in a knowledge-based system is outlined.

The study shows that it is feasible and reasonable to incorporate both random and nonrandom uncertainties encountered in survivability and vulnerability assessment. Fuzzy models can be used to model uncertainties which, for the most part, have been ignored or dealt with implicitly throughout judgment. This study suggests how fuzzy models may be used to augment and improve current technology.

PREFACE

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ILLUSTRATIONS

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I. INTRODUCTION

BACKGROUND

Survivability and vulnerability (S/V) assessment of protective structures involves many uncertainties. Most of the uncertainties arise because of complexity in the weapons effects phenomena and limited data base. Evaluation and analysis of S/V are further complicated by inherent variability (e.g., material behavior, threat scenario, explosive effects) and imprecision (e.g., definition of damage, failure) in every component of the assessment chain.

The importance of uncertainties has always been recognized in S/V assessment. However, until very recently, the treatment of uncertainties is limited to statistical techniques. Uncertainties are modeled exclusively as random variables. A parameter is assumed random when it is uncertain. This approach is used to treat uncertainty in the analysis models as well.

RANDOM VS NONRANDOM UNCERTAINTIES

The study described in this report is based on the belief that not all uncertainties encountered in S/V assessment are random, and that the different kinds of uncertainties should be modeled and analyzed by using different but appropriate procedures. To review briefly, random variables are based on the concepts of probability (viz., a definite sample space, repeatable experiments under near-identical conditions, and meaningful sample average and variance). On the other hand, there are many uncertainties in S/V assessment which have quite different characteristics, viz., the sample space is not well defined, there are few samples, and the sample average and variance may not be meaningful. In particular, linguistic data, subjective judgment, imprecise information, and extreme complexity or details are examples of uncertainties which should not be considered random.

SCOPE OF STUDY

The study seeks to extend current statistical S/V assessment
capabilities in two major directions: (1) Within the confines
of random models of uncertainties, to explore how stochastic
methods can be better utilized; and (2) When the uncertainty is
nonrandom in nature, to explore ways to model and analyze the
uncertainty, and finally to incorporate it into the overall S/V
assessment.

The starting point of the study is a critical review of
current statistical assessment approach, and the ways by
which it can be expanded and improved. This is summarized in
Section II. It is argued in Section III that at least two types
of uncertainties should be recognized, although there may be
more. Uncertainties that can be modeled as random are distin-
guished from uncertainties which are not related to random occur-
rence. In this study, all nonrandom uncertainties are con-
sidered to belong to one broad category called fuzzy uncer-
tainties which can be modeled by fuzzy sets and fuzzy logic.
Suggestions for use of fuzzy uncertainties in S/V assessment and
how they can be integrated with random uncertainties are given.

Important extensions to current assessment methods based on
this view are summarized in Section IV, with details given in
Sections V and VI. Random variable and random process models of
uncertainties are the subject of Section V. Types of random
equations and their solution techniques are described. Emphasis
of the study is on numerical methods, or numerical stochastics,
since it is well-known that analytic solutions to random equa-
tions are difficult to obtain. The study includes a study of
stochastic difference equations, and other discretization methods
for random equations.

Section VI summarizes two methodologies to solicit and
aggregate expert opinions, a vital element of the assessment
process which is viewed as nonrandom. Opinions on the perfor-
mance of an analytic model, expressed in terms of gravity and
effect, are used to construct fuzzy relations which then repre-
sent the uncertainties regarding the model. These opinions,
together with other available data (deterministic, random or
otherwise) in the assessment of structural damage, can be imple-
mented within the framework of a knowledge-based system.

This report emphasizes engineering description of the problem, whether the discussion is on S/V assessment or the modeling and analysis of uncertainties. Mathematics are kept to a minimum, and only the necessary background on stochastic equations and fuzzy sets is included. For details on the mathematical theories and development of the examples described in this report, the reader is referred to the publications and technical papers cited.

II. SURVIVABILITY AND VULNERABILITY ANALYSIS
OF PROTECTIVE STRUCTURES

The current procedure used in S/V analysis of protective structures is described. The uncertain information and data which must be correctly incorporated into the analysis are delineated. The current probabilistic approach in handling system and other uncertainties and its inadequacies are discussed. The inadequacies are caused mainly by the failure of current approach to recognize that not all uncertainties can be adequately modeled as random variables.

TYPICAL PROBLEM

The S/V analysis of protective structures is historically a complex civil engineering problem which can be approached only by assessing a number of components of the problem. Many of these components involve uncertainties. For example, consider the survivability of a known buried-box structure under the influence of an assumed extreme environment, specified in terms of the mechanical effects of airblast and ground shock. The analysis is approached by formulating the problem in the following components (see Fig. 1):

(1) Identify the important failure modes and their associated fragilities;

(2) Identify the explosive effects and loading mechanisms responsible for these failure modes, and the uncertainty in the environment description;

(3) Determine the response function (i.e., the relationship between the loading and fragility parameters) and its associated uncertainties; and

(4) Compute the system failure probability by comparing the response and fragility for each failure mode of each component, and by combining the component survivability probabilities into the system survivability probabilities.
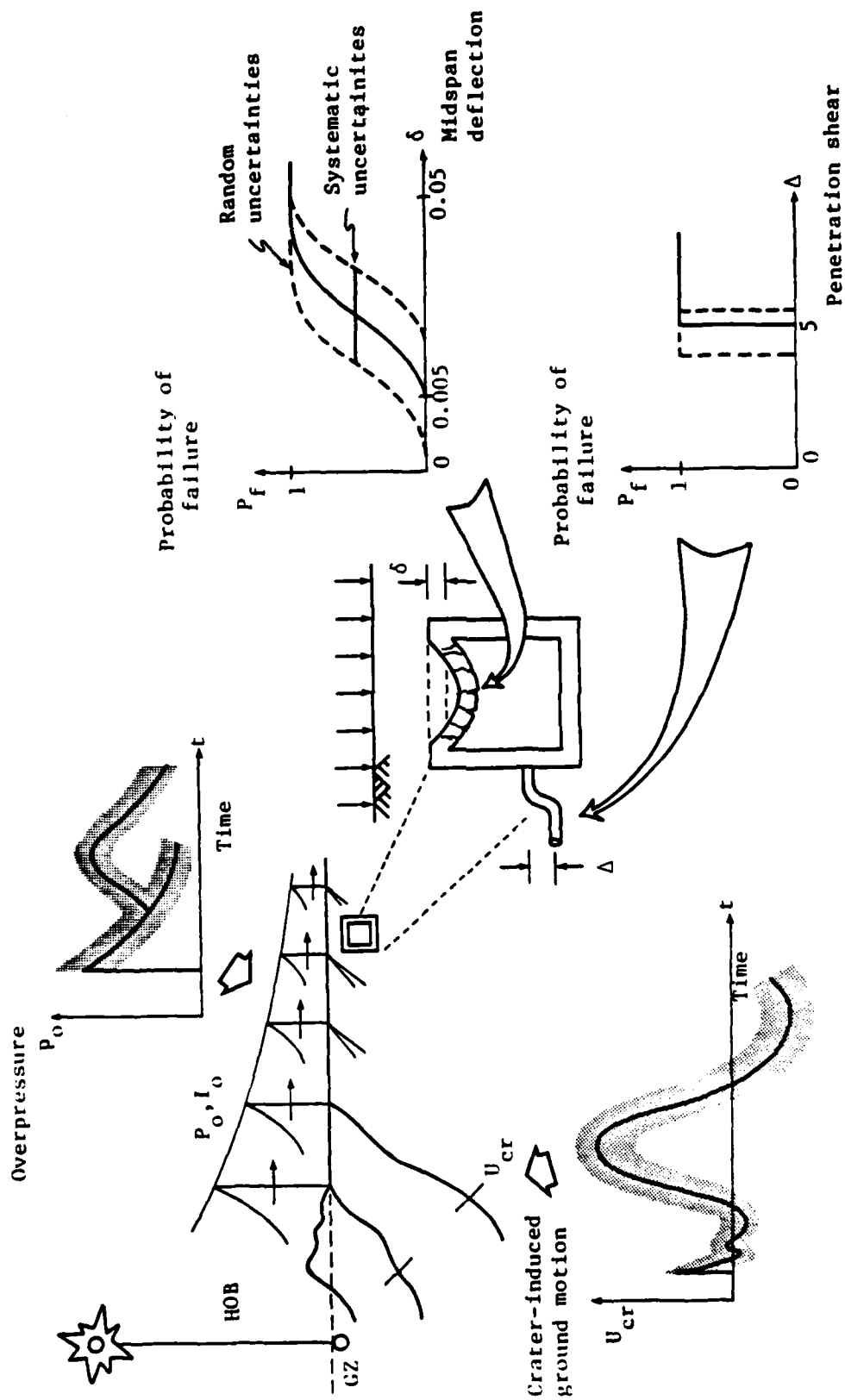
Figure 1. Typical protective structure analysis.

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TYPICAL UNCERTAINTIES

For the buried-box example, a partial list of the information and data required in each step of the analysis can be summarized in Table 1. The information is further divided into three categories: deterministic, probabilistic and others. Such classification is not common practice, but is done here for reference in subsequent discussion. The three categories correspond to decreasing precision in the information. Current methodologies acknowledge only the first two categories, i.e., deterministic and probabilistic. Very imprecise information, subjective judgment and linguistic data are treated as deterministic or probabilistic, or ignored altogether. For example, cratering and its associated effects, nonideal surface effects, the effects due to the choice of the soil-structure interaction model, and the damage evaluation criterion, to name a few, are treated as deterministic or vaguely as a source of probabilistic uncertainty called systematic uncertainty.

Table 1 describes a fairly complex process where imprecision exists, even in the identification and classification of the sources of uncertainties in the protective structure analysis. Furthermore, any tabulation of uncertainties, such as those shown in the table, is very subjective. Another person of similar background and experience will most likely arrive at a different tabulation. This possibility serves to underscore the present state-of-perception of the problem.

CURRENT TREATMENT OF UNCERTAINTIES

In an analytic treatment, it is assumed that the total uncertainty, Ω, consists of the uncertainty due to inherent randomness, ε, and the uncertainty associated with the error in the prediction, Δ, such that (e.g., see Ang and Tang, Ref. 1)

$$\Omega^2 = \varepsilon^2 + \Delta^2 \tag{1}$$

Hence, for a cause-effect relationship such as

$$R = g\ (X_1,\ X_2,\ \ldots\ldots,\ X_n) \tag{2}$$

TABLE 1. PARTIAL LIST OF UNCERTAINTIES

Source → Loading → System Dynamics → Critical Response → Survivability Assessment

Survivability Assessment:
- Interpretation of Response
- Failure of Damage Criterion
- Probability of Survival
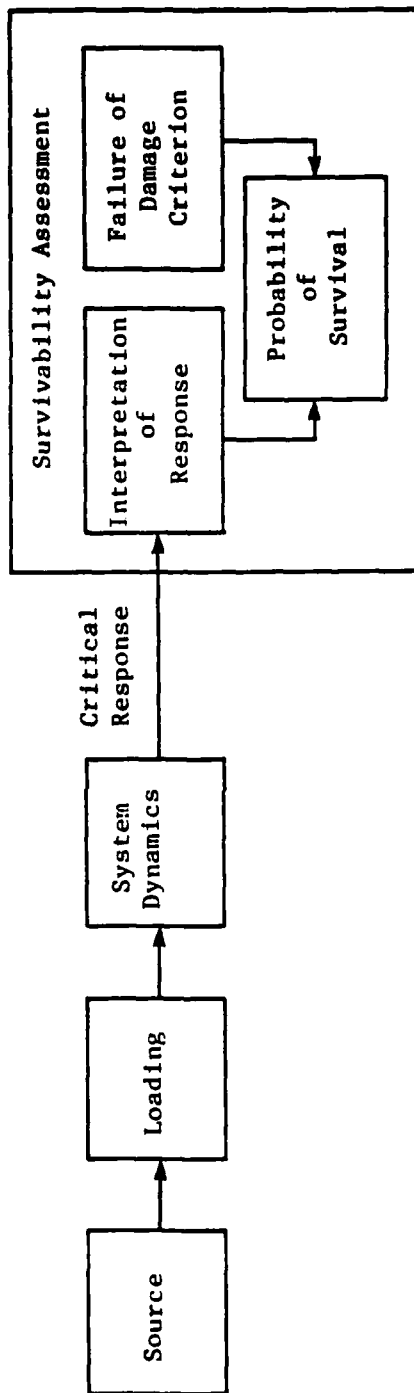
	Source	Loading	System Dynamics	Survivability Assessment
Deterministic	Yield Source Type	Ideal Airblast	Configuration Network	Failure Modes
Probabilistic	HOB CZ		Material Properties	Component Capacity
Others	Source Type	Cratering Ground Shock Nonideal Airblast Airburst	Interaction Model Structural Models	Damage Failure Impact on System

where R is the dependent variable and X_i, $i = 1,2,\ldots\ldots,n$, are the independent variables, the uncertainty in R can be evaluated through first-order analysis. The mean value of R is

$$\bar{r} = g\ (\bar{x}_1,\ \bar{x}_2,\ldots\ldots,\ \bar{x}_n) \tag{3}$$

where $\bar{x}_i$ denotes the mean value of x_i and the variance of R is

$$\sigma_R^2 = \rho_{ij}\ c_i\ c_j\ \sigma_{X_i}\ \sigma_{X_j} \tag{4}$$

where ρ_{ij} is the correlation coefficient between X_i and X_j, $c_i = \partial g/\partial X_i$ evaluated at $\bar{x}_i$, and $\sigma_{X_i} = \Omega_{X_i}\ \bar{x}_i$. Alternatively, the random and systematic uncertainties are propagated separately to give the random and systematic uncertainty in the dependent variable R, as described in a survivability assessment procedure by Rowan (Ref. 2).

Monte Carlo schemes also have been devised to treat more complex survivability problems based on the same principle. Collins (Ref. 3), for example, used the two-tiered sampling approach illustrated in Fig. 2 to distinguish between random and systematic uncertainties. A limited number of inner loop samples is taken to estimate the average survival probability due to random variations. Each outer loop sample results in an estimate of the median value of the average survival probability. The outer loop estimates can then be used to assess the range of variation of the median probability and establish the confidence levels.

Whether the analytic or Monte Carlo method is used, the result of such an analysis can be illustrated in Fig. 3. The median probability of failure curve gives the probability of failure of the system for a particular loading, and is often referred to as the fragility curve. The curve is a result of the incorporation of random uncertainties only. The variability of this curve when systematic uncertainties are taken into consideration is also illustrated in the figure. The lefthand and righthand bounds normally correspond to the 10 percent and 90 percent confidence levels that the probability of failure will be between these bounds in spite of uncertainties in the analysis model, imprecise data, and so forth. In some works, such as those by Wong and Richardson (Ref. 4) and Rowan (Ref. 2), modeling errors are treated as a source of systematic uncertainties.
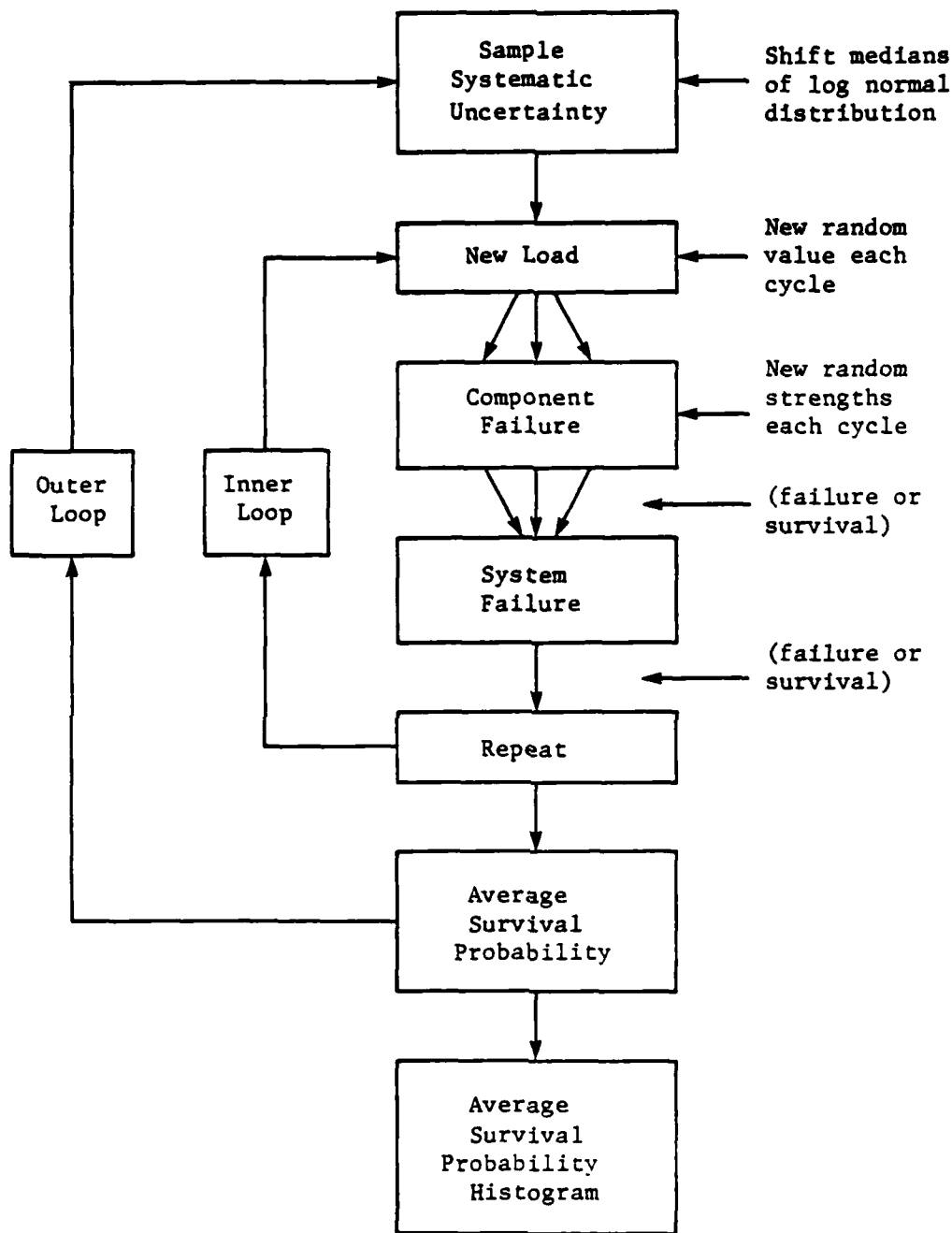
Figure 2. Two-tiered Monte Carlo sampling procedure.
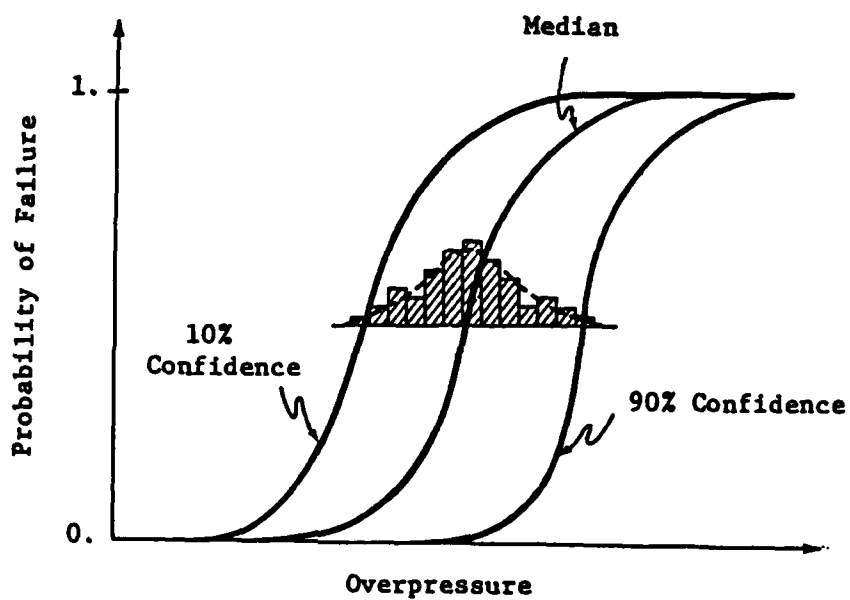
Figure 3.   Typical fragility curves from a survivability analysis.

INADEQUACY OF CURRENT PRACTICE

The analysis procedure described above is very appealing, since it produces a precise summary of the different effects of uncertainties on the system response--for example, in the form of Fig. 3. Such information can be fed into the hierarchy of decision-making in strategic and tactical survivability analyses. In actuality, the separation of random and systematic uncertainties is not a simple task, if possible at all. The unilateral treatment of all uncertainties by probabilities implies many assumptions. For example, it assumes that the data base exists, that random and systematic uncertainties are independent, and that all types of systematic uncertainties (such as biases, judgment and modeling error) are similar and can be treated in the same fashion. These assumptions are seldom justified due to very limited data, lack of knowledge, and incomplete understanding of the complex physical phenomena and structural behavior.

To illustrate, consider one particular step in the analysis procedure shown in Fig. 4, which depicts the choice of a model for the dynamic response of the roof of a buried box due to soil-structure interaction loading. Many models can be considered, including a single-degree-of-freedom model commonly used in slab analysis, a multiple-spring-mass model, or a finite element model of the complete soil-structure configuration. Subjective judgment enters into such a choice, and the uncertainty associated with the choice (i.e., the computational error to be expected) is assessed from past experience with similar computations and comparative analyses. Furthermore, the evaluation of modeling uncertainties relies heavily on comparing model predictions with results from controlled tests, which are seldom feasible.

Ignoring these concerns for the time being (they are addressed in Wong and Richardson, Ref. 4), suppose that the finite element modeling approach is selected. One is then faced with more modeling decisions--for example, the choice of a soil model, a concrete model, the method to incorporate reinforcing steel, the size of the finite element mesh, and so on. Another layer of judgment and decision is encountered. To proceed, let us limit the discussion to the modeling of the roof-slab of the box. One can consider using a composite reinforced-concrete model where the reinforcement is smeared over the volume of the finite element, or an explicit reinforcing steel model, or a model comprised of two or three elements across the thickness of the slab,
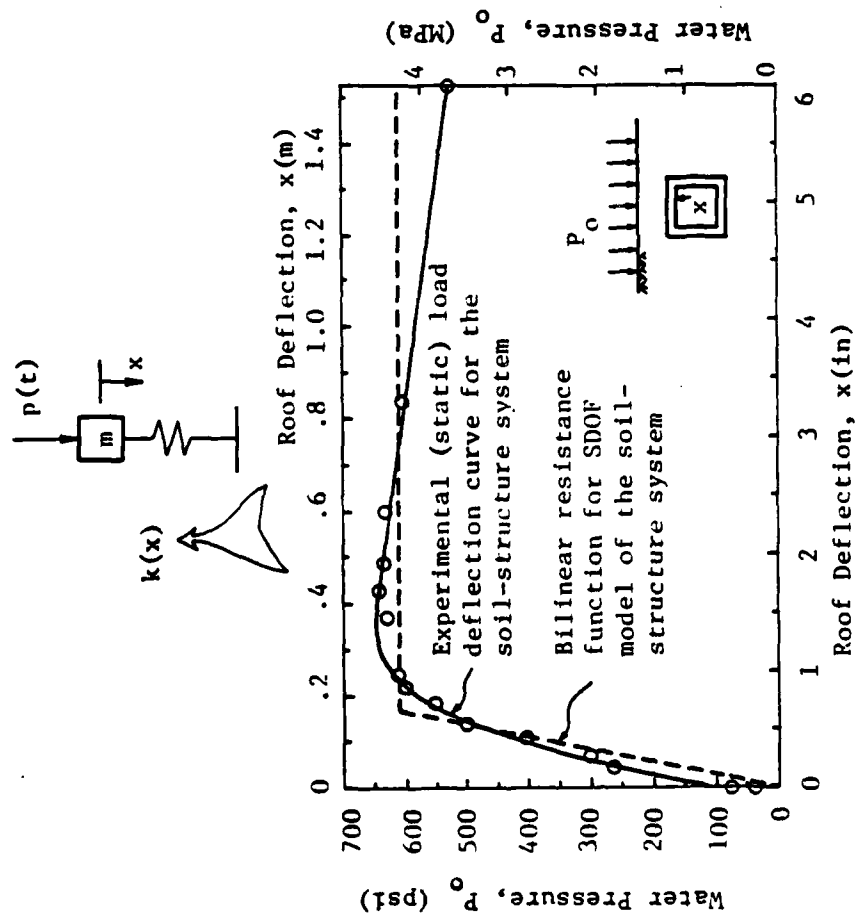
(b) Multiple degree-of-freedom model
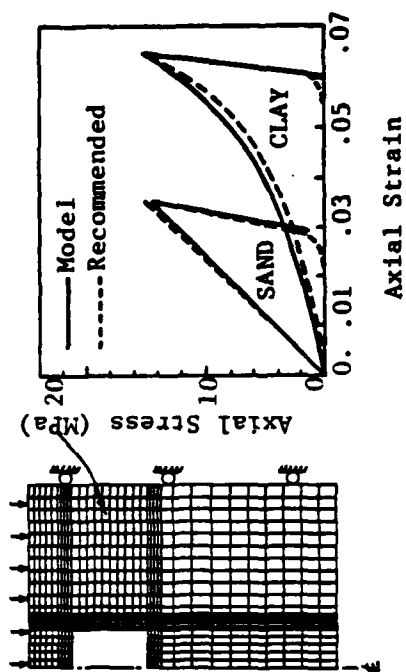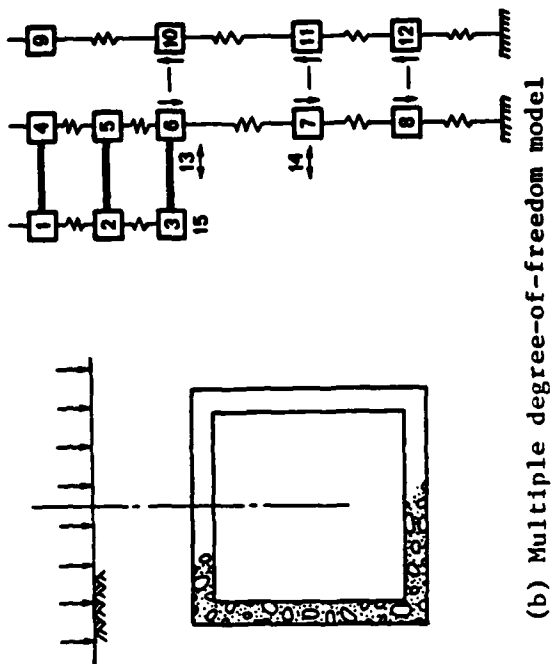
(c) Two-dimensional finite element model

(a) Single degree-of-freedom model

Figure 4. Possible dynamic soil-structure interaction models for survivability analysis.

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and so on. Details of modeling notwithstanding, the fact is that much subjective judgment and recollection of past experience are used in the analysis. More important, these data cannot be summarized readily by a probability density function.

As an illustration, the impacts on the response of the roof slab due to the modeling options mentioned previously are summarized in Fig. 5. The figure shows the variation in the roof-slab velocity and the loading exerted by the soil on the roof slab of the buried box as a result of these modeling assumptions. This variation will be referred to for the time being as the modeling or systematic uncertainty associated with the slab model. It is apparent that such uncertainty is not amenable to a probabilistic description. Furthermore, experience shows that the explicit rebar model gives fairly good results in the slightly plastic range, that the same model is less accurate in the membrane tension mode, and that the shear failure modes are absent from such a model. These are important data which should be incorporated into a realistic analysis. Such inputs are ignored in the current probabilistic formulation, since they cannot be readily assimilated in the probabilistic description of uncertainties.

More examples of this type will be given in the next section. For a detailed description of the inadequacies of current practice based on the all-probabilistic modeling approach, see Wong, Ross and Boissonnade (Ref. 5). To summarize, by acknowledging random models as the only models of uncertainties, current formulation of S/V assessment methods either ignores nonrandom uncertainties altogether (since they cannot be assimilated into the current framework) or forces them to be random as well.

(a) Midspan slab velocity

(b) Midspan interaction loading
    on roof

- - - Elastic structure

[shaded box] Sensitivity to
structural modeling
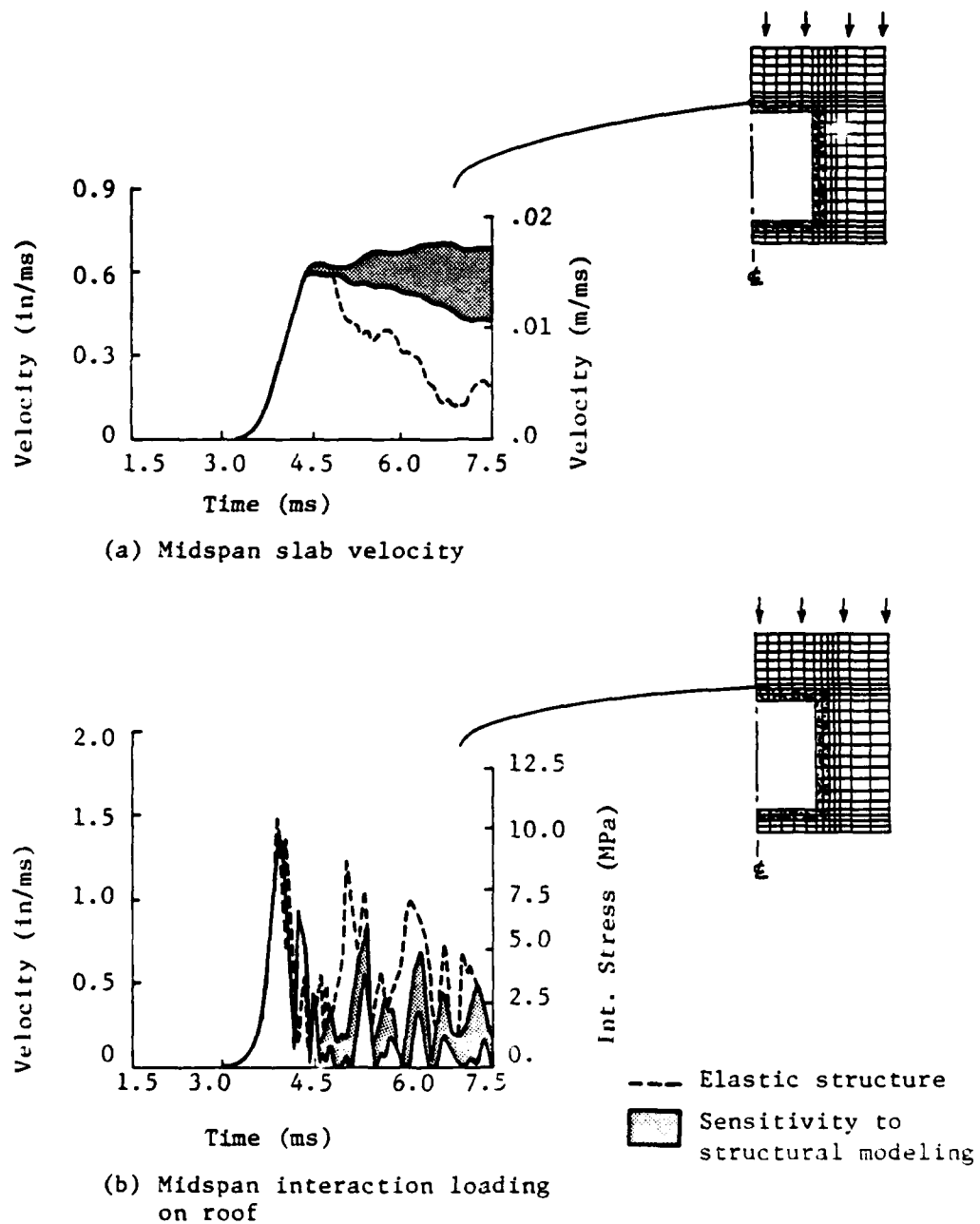
Figure 5.   Sensitivity of interface load on buried box and roof
            slab to uncertainties in structural modeling.

III. UNCERTAINTIES

To researchers in the S/V assessment community, there is probably no need to espouse the importance of uncertainties. However, the term uncertainty may have different meanings to different people. Several examples of uncertainties were mentioned in the previous section. The discussion is continued here and the variety of uncertainties encountered in S/V assessment is further illustrated to support the simple classification scheme that is proposed. Uncertainties are divided into two groups, according to whether they can be modeled adequately as random variables or not. The need to integrate random and non-random models of uncertainties into an overall assessment framework is also addressed.

SOURCES OF UNCERTAINTIES

The term uncertainty may be associated with ambiguity, fuzziness, randomness, vagueness, imprecision, subjectivity, or extreme complexity. For a more detailed description of these terms, see Yao and Furuta (Ref. 6). In essence, uncertainty arises because one is not sure about the outcome of a real-world event, and in an effort to understand this event, he postulates a concept or model of the real world.

In the context of structural mechanics, the main sources of uncertainties may be delineated by considering the analysis chain, as shown in Fig. 6 (after Blockley, Ref. 7). The basis of structural mechanics is Newtonian mechanics in the form of conservation laws. These laws are used together with energy methods, virtual work methods, and so on, to formulate the governing equations of motion. Material behavior is incorporated through constitutive models. Structural behavior is obtained by solving the equations, and by making more assumptions on the boundary conditions, the loading and the solution procedure. These assumptions are an integral part of the modeling process. Finally, behavior of a prototype structure or future structure is inferred from the analysis result after it has been assessed and evaluated with reference to one's experience with similar structures and one's knowledge of structural behavior in general.

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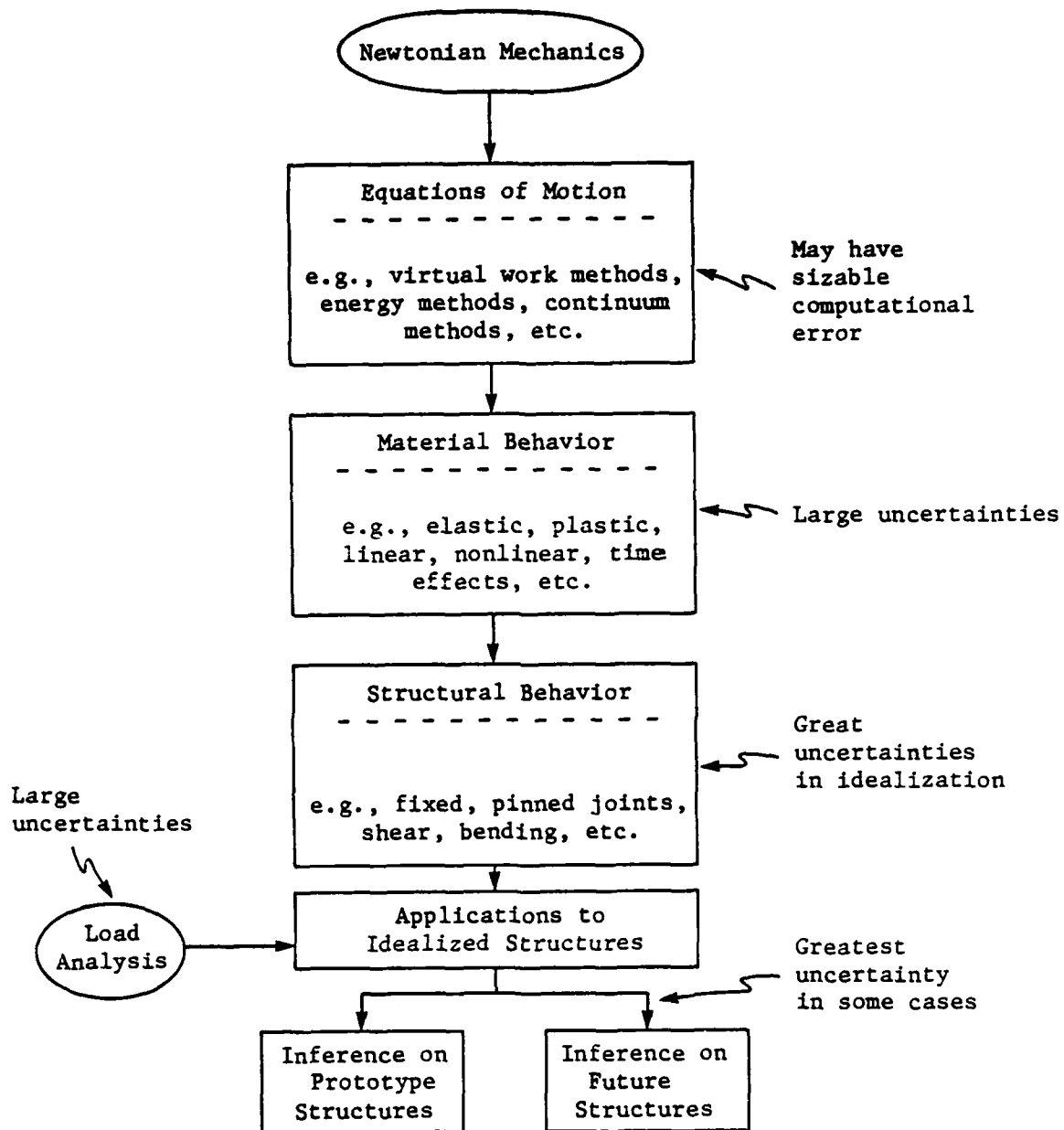
Figure 6. Analysis chain in structural dynamics.

Uncertainties of all descriptions may enter into every step of the analysis procedure. Moreover, they also appear in various degrees. Some are judged less important than others and, hence, neglected. Jugdment is itself a source of uncertainty. For example, Newtonian mechanics is, strictly speaking, incorrect; relativistic mechanics is known to be better. However, engineers find that for most of their work the former is adequate. Here lies the major difference between science and engineering: Mathematical rigor or truth is not the only criterion in modeling. It is what makes a good engineering model that counts. The application of judgment is also evident in the choice of the solution procedure, the material model and end-constraints. It allows the engineers to simplify the complex problem, focus on the important issues, and arrive at an engineering solution knowing full well that such a solution may not be perfect and that some compromises have been made along the way.

UNCERTAINTIES AND MODELING

Hence, one approach to classify the uncertainties in S/V assessment is to consider the assessment as a modeling problem. The types of uncertainty are related to the selection and definition of the model. This is the approach used here.

In modeling, the amount of data supporting the model is an important issue. With a large data base, the choice of the model is more clear-cut; the model can be better justified. A deterministic model is justified when variation about the nominal is negligible or inconsequential. A random model is used when variation is not negligible, but can be determined to follow a certain probability distribution. When the data base becomes smaller, the choice is less clear, and the situation becomes more fuzzy. Note that even in this simple view, the distinction between deterministic and random, and between random and others, is itself not well-defined, but relies heavily on judgment and knowledge. Nevertheless, a rationale based on the size of the data base seems reasonable and is readily accepted by most engineers.

In current assessment methods, only the deterministic and random categories are recognized. The category designated as others is either ignored or included as the so-called systematic uncertainty or bias, and is considered to have random

characteristics as well. Despite the fact that the same mathematical model is used for both categories (random and systematic), the latter can be easily detected in current procedures. Systematic uncertainties are operated upon separately from random uncertainties, and the probabilistic description of systematic uncertainties is often determined in an _ad hoc_ or arbitrary manner.

However, there are uncertainties which are nonrandom in nature, the size of the data base notwithstanding. Some examples are fuzziness resulting from subjective knowledge, ambiguity inherent in linguistic descriptions, and vagueness associated with ill-defined figures and pictures. Although engineers encounter these uncertainties almost on a daily basis, and handle them with ease using their experience and judgment, the fact remains that these uncertainties have not been represented explicitly in S/V analysis.

NONRANDOM UNCERTAINTIES

This discussion starts with random uncertainties, since they are familiar to engineers, and then leads into nonrandom uncertainties. Differentiation between random and nonrandom quantities is made mainly by reference to examples taken from S/V assessment. Formal classification is given in the next subsection.

Uncertainties in basic properties--A typical random uncertainty is that associated with material properties, such as the strength of concrete shown in Fig. 7. A large number of core samples which are nominally the same are tested under near identical conditions. The result is represented by the histogram in the figure which can be approximated by a (probability) density distribution. While the fit between the theoretical distribution and the experimental data may not be perfect, the random model captures the variation in the strength well enough for engineering purposes. Hence, one says that the uncertainty in the concrete strength is mainly random, and can be modeled by a random variable.

Note that implicit in this modeling process is the assumption that the effect due to the choice of the distribution (normal, lognormal, etc.) is negligible compared to the variation in the quantity of interest, viz., the strength. The selection of

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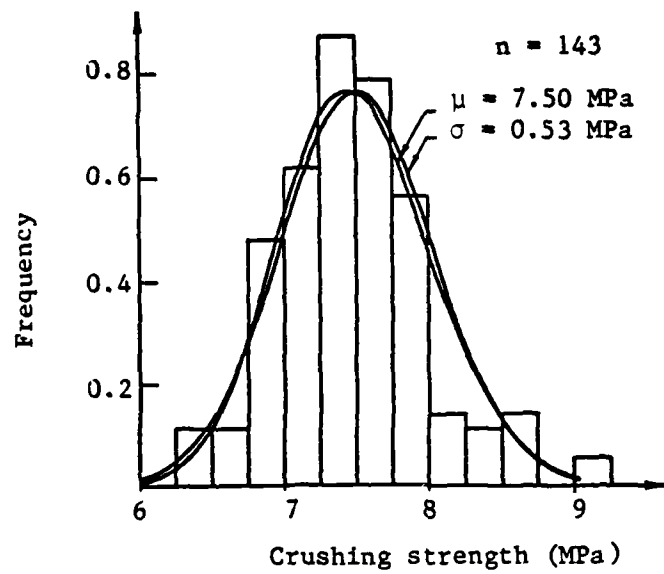
Figure 7. Frequency diagram of crushing
strengths of concrete cubes.

one distribution over another is a nonrandom act, although in
this case the latter issue is unimportant since the effect on the
overall modeling is inconsequential.

Consider now Fig. 8, which shows the variation in concrete
strength under biaxial loading (Ref. 8). Seven well-known insti-
tutions in the U.S. and Europe were asked to perform strength
tests on nominally identical concrete samples in order to esti-
mate the effects due to random variation, sample geometry and
test machines. Figure 8 is representative of the results ob-
tained. Differences in strength due to differences in sample
geometry and testing technique can be inferred from the differ-
ences in the solid lines in the lower half of the quadrant. For
a particular solid line, variations due to inherent randomness in
material properties are shown by the shaded areas in the upper
half of the quadrant.

The uncertainty represented by the shaded areas correspond
to those shown in Fig. 7, and can be represented by probability
distributions. This is not done to avoid adding more confusion
to the figure. On the other hand, appropriate treatment of the
uncertainty indicated in the lower half is less obvious. However,
based on the data given in the figure, it is difficult to justify
treating this variation as random. Furthermore, note that the
spread of the data in the lower half of the quadrant is much
larger than the spread within a particular shaded region in the
upper half of the quadrant, which corresponds to random uncer-
tainty. Hence, contrary to the situation in Fig. 7, random
uncertainty is small compared with the nonrandom uncertainty in
this case. If fact, all the shaded regions in Fig. 8 can be
ignored, and the variation in strength still can be represented
well by the data in the lower half of the quadrant.

What is shown here is that elusive uncertainty, currently
called systematic uncertainty or bias. The main question is
whether such uncertainties can be represented as random vari-
ables. Said another way, when data of this type is considered
random, what are the physical meanings of the sample expectation
(mean) and sample variance? These questions abound in S/V
assessment and are not limited to structural properties. Figure 9
shows a similar set of data for soil, and Fig. 10 summarizes
typical variations in the blast pressure from a high explosive
experiment.
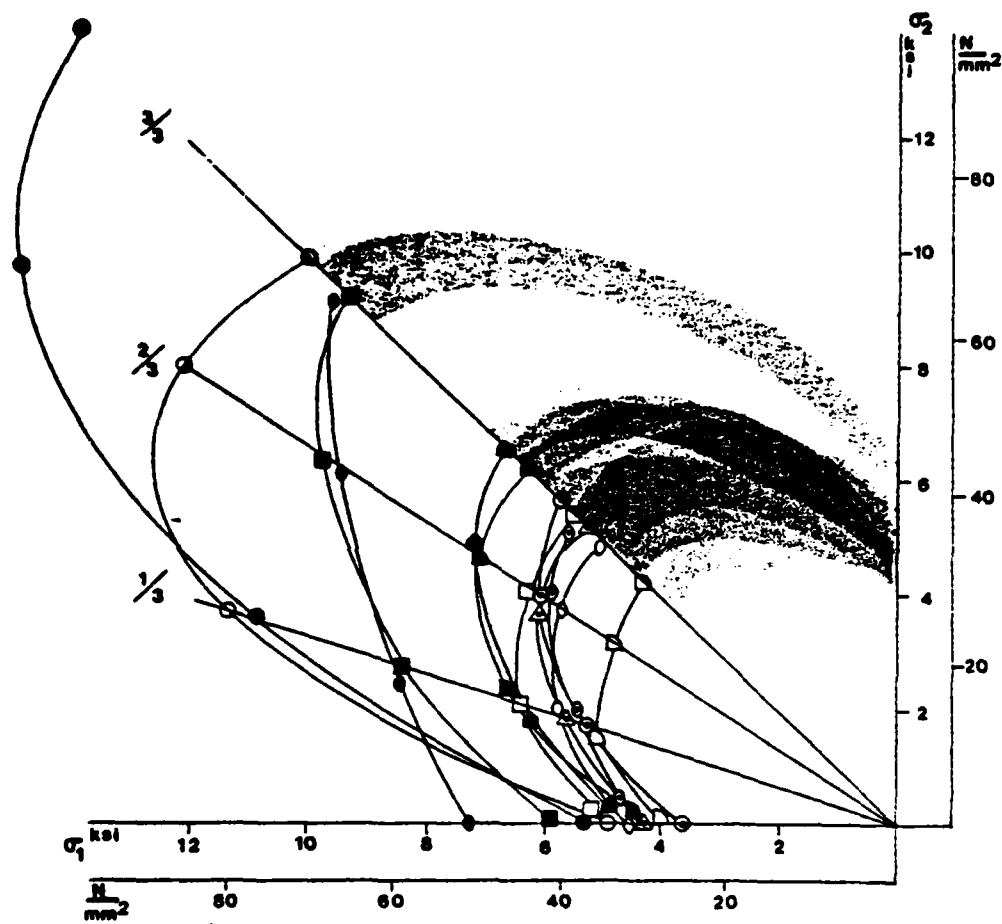
Figure 8. Strength of concrete under biaxial loading.

Axial Strain (%)

Axial Stress (MPa)

(b) Undisturbed test specimens

Axial Strain (%)

Axial Stress (MPa)
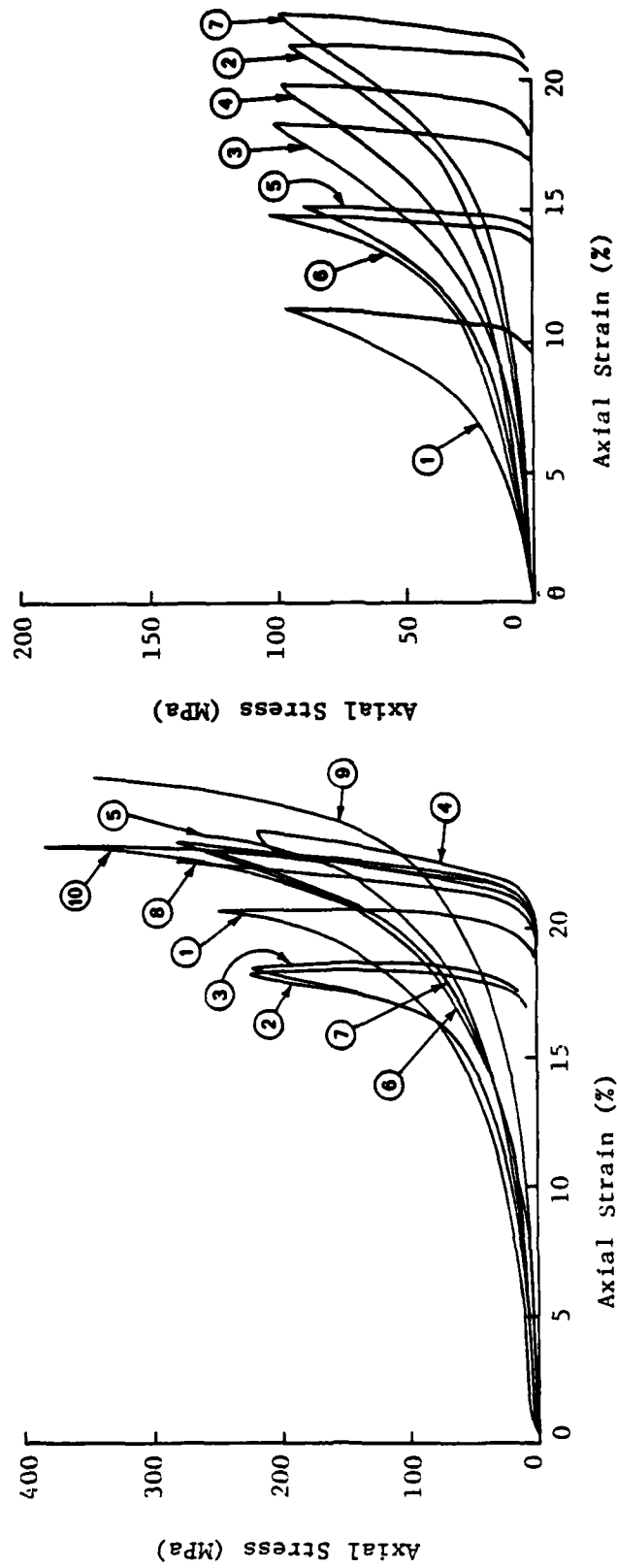
(a) Remolded test specimens

Figure 9. Typical comparison of dynamic UX test results for remolded and undisturbed specimens of sand.

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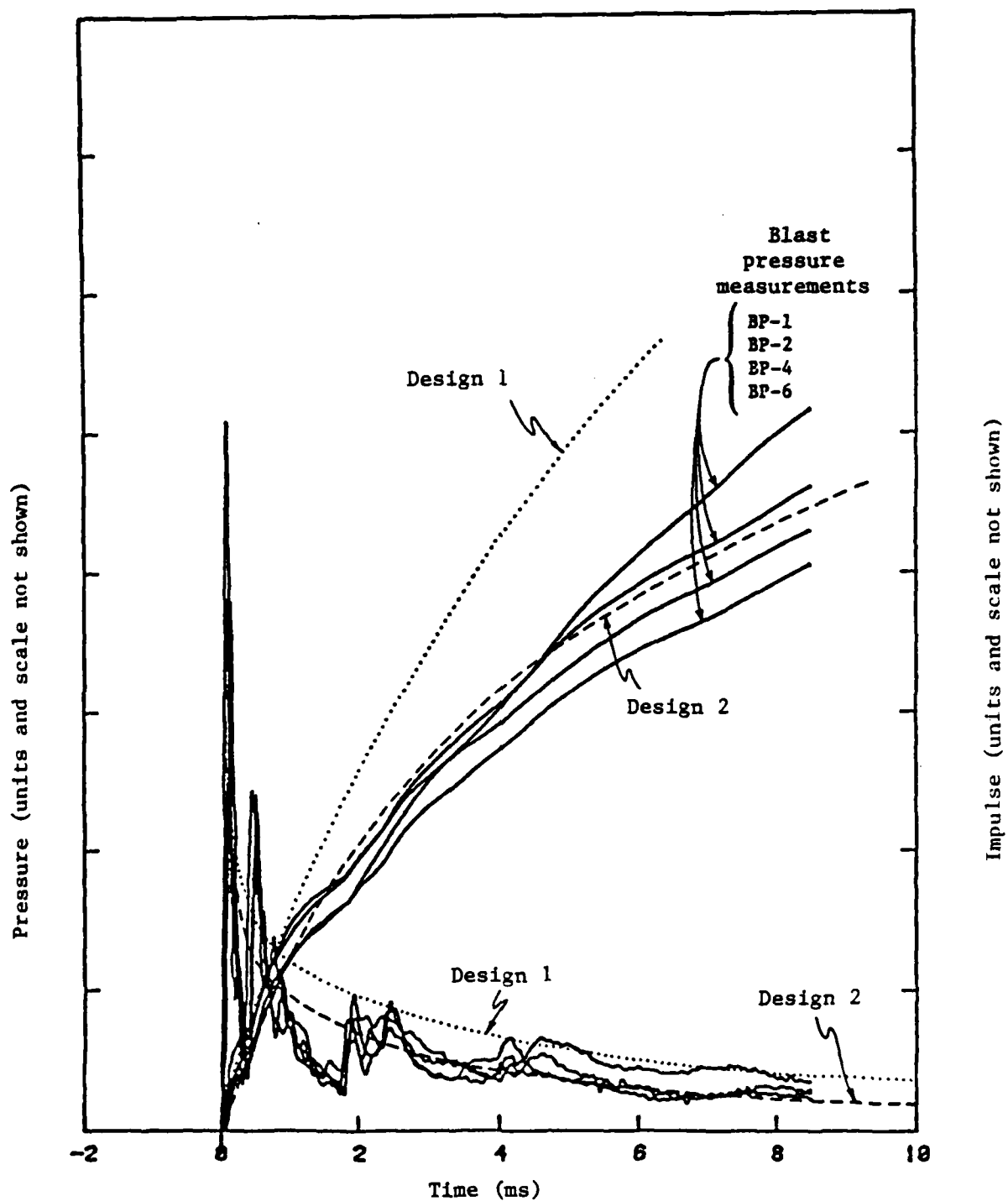
Figure 10. Typical comparison of blast pressure measurements and analytic waveforms.

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Uncertainty in system response--Knowing the uncertainties in
the basic properties does not mean that uncertainties in the
system response are known. In the previous discussions, uncer-
tainties introduced by testing instruments are mentioned. Mea-
surement transducers can and do introduce further uncertainties
in the measured properties, such as those shown in previous
figures. This is also true in a larger scale, such as in a field
test. Typical variations in the measured soil stress, such as
shown in Fig. 11, can be due to variations in the loading, soil
properties, and measurements. The task facing the engineers is
how to infer the physical phenomenon from these data, and its
associated uncertainty. The challenge has been met largely by
choosing the average, which, of course, is the same as consider-
ing all variations to be random.

Uncertainty in simulation models--In S/V simulations, a
model of the system (real-world) is postulated. Parameters of
the model are assigned based on available data, e.g., the
material properties described previously. The equations are
solved to give a solution. The solution is deterministic if all
parameters are deterministic. The solution is random if at least
one parameter is random. The latter is basically the current S/V
assessment procedure. Note the system model itself is always
deterministic, and the procedure can be considered in more gen-
eral terms as the propagation of statistical uncertainties (when
parameters are random), as shown in Fig. 12.

When another model which may involve a different set of
parameters is used, a different response which can also be random
may be obtained. An example in the context of soil-structure
interaction was given in the previous section. If one considers
several random responses obtained by using different models, what
use can one make of them? This challenge has not been met.
Taking the average is not the answer, as is obvious from the
soil-structure interaction example. This is because the uncer-
tainty associated with modeling is nonrandom in most cases.

Uncertainty in model evaluation--Uncertainty in a model is
connected with how closely the model represents the real-world.
This is the task of model evaluation. Typically in the
laboratory, models are evaluated by comparison with controlled
experiments. An example on the strength of columns subjected to

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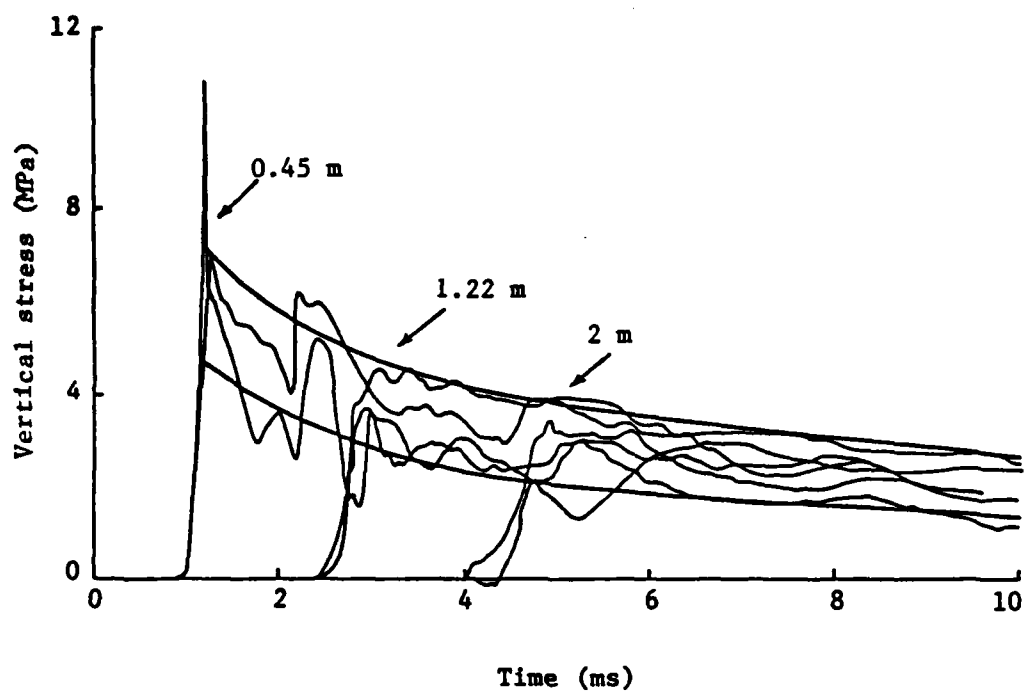
(a) vertical stress
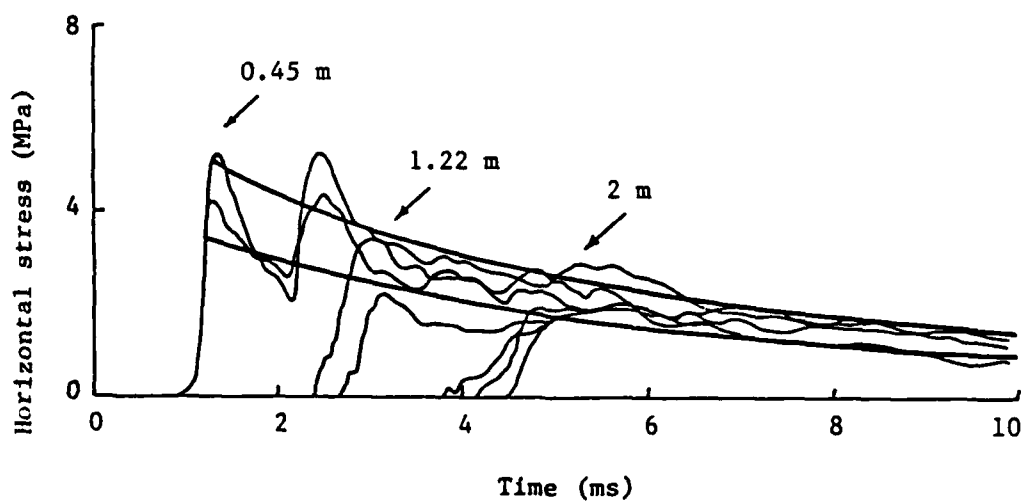
(b) Horizontal stress

Figure 11. Typical free-field soil stress measurements at several depths.

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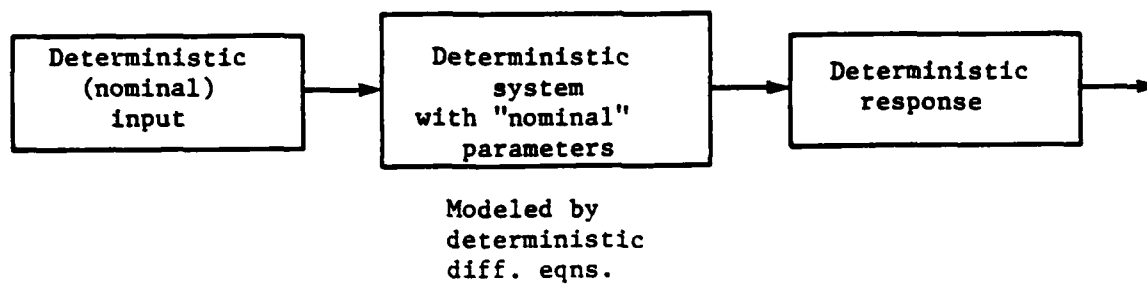
(a) Deterministic approach
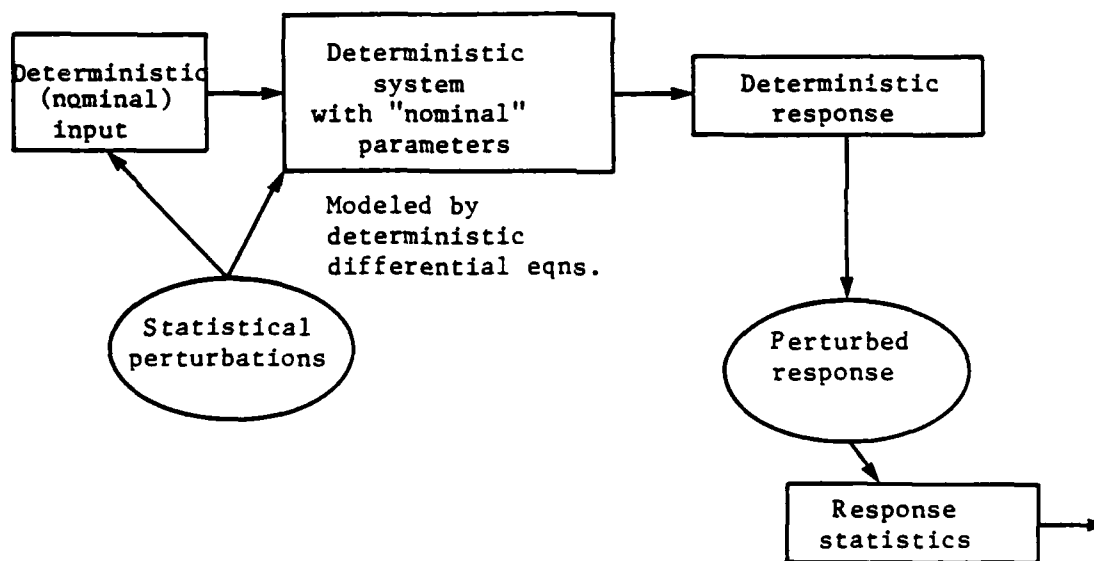
(b) Statistical approach based on deterministic
    model

Figure 12. Statistical approach based on parametric perturbations.

combined bending and axial load is shown in Fig. 13. The experimental data based on twenty samples are compared with the theoretical model, which in this case is the ACI/NBC design equation for short columns. Presented with this fact, one can say that the column model is some 15 percent conservative compared with the experimental results. This conservatism can be corrected, by a scale factor for example. The uncertainty in the corrected model is then in the shape of the spread and the generality of the correction factor for other similar columns. Note this uncertainty is not random.

Basically the same procedure is followed to evaluate models of field events. Figure 14 shows the standard approach to evaluate a nominal concrete or soil model. Since this is common and familiar practice, there is no need to go into details and only the main points will be elaborated upon. First, field tests and field data are much more scarce than laboratory tests and data. Second, the data now involve more uncertainties. Factors which can be controlled in the laboratory may not be controlled in the field. In situ measurements are more difficult and less reliable than in the laboratory. Inference based on comparison of model response with one or two field measurements is very different from inference based on the data shown in Fig. 13. A representative comparison which shows good correlation between test and model responses is given in Fig. 15. Often, the agreement is not as good, such as shown in Fig. 16. Even in the case of good agreement, however, the meaning of good, and how such goodness can be used to improve the model remain fuzzy.

Judgment and other uncertainties--Other important elements of S/V assessment which have not been discussed so far are judgment and subjective opinions. These are uncertainties because one person's judgment may be different from another. The source of difference can be attributed to subjectivity, experience and knowledge, but yet cannot be isolated. Linguistic terms often used in engineering evaluation have generally accepted meanings, but are equally vague. Finally, structural data on which the assessment is to be based includes a vast amount of photographs, drawings, and so on. These soft media carry a great deal of information but it is difficult to translate the information.
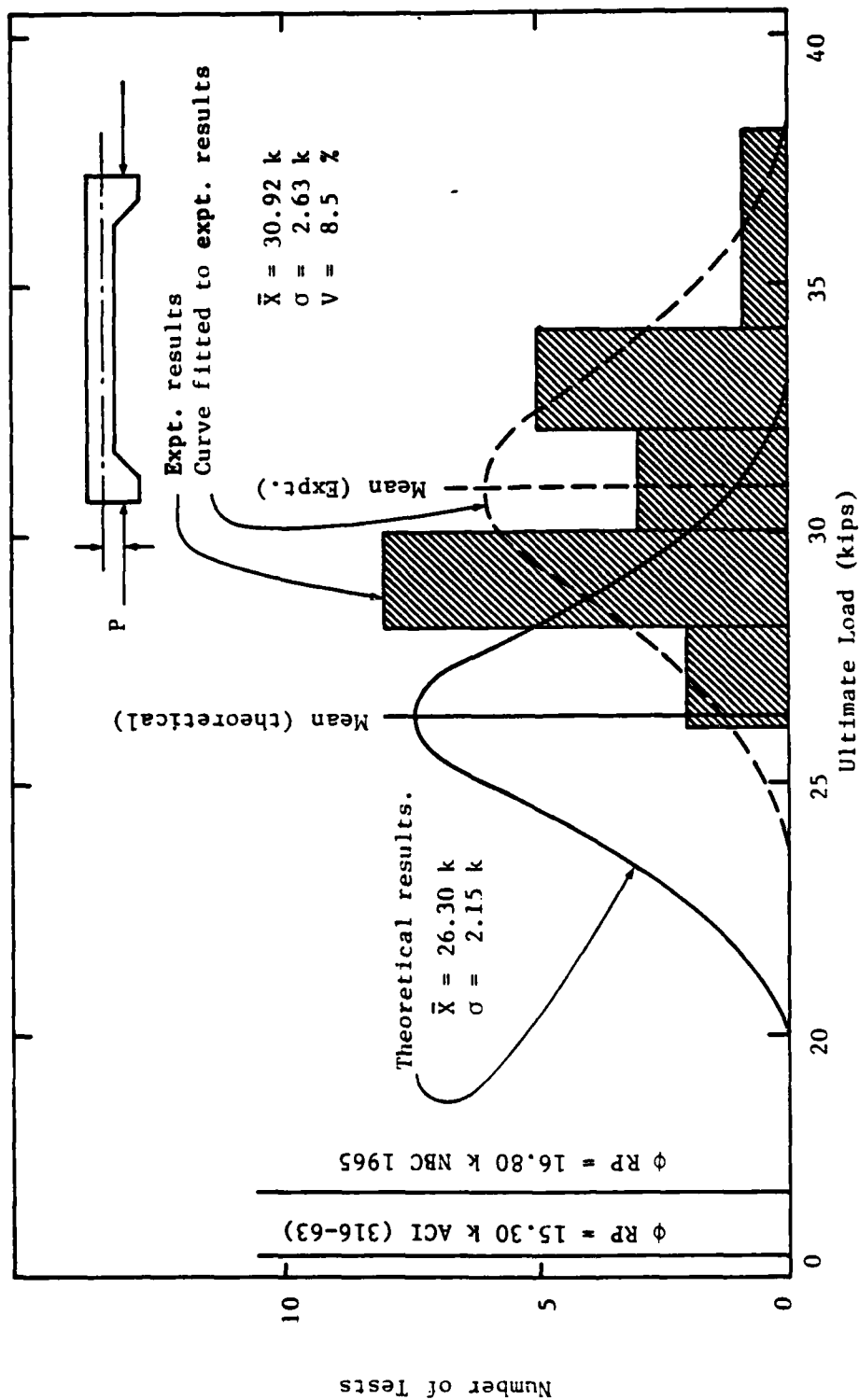
Figure 13. Comparison of experimental and theoretical distribution for column.
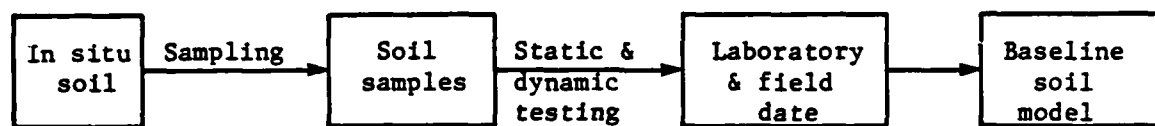
(a) Determinination of baseline soil model
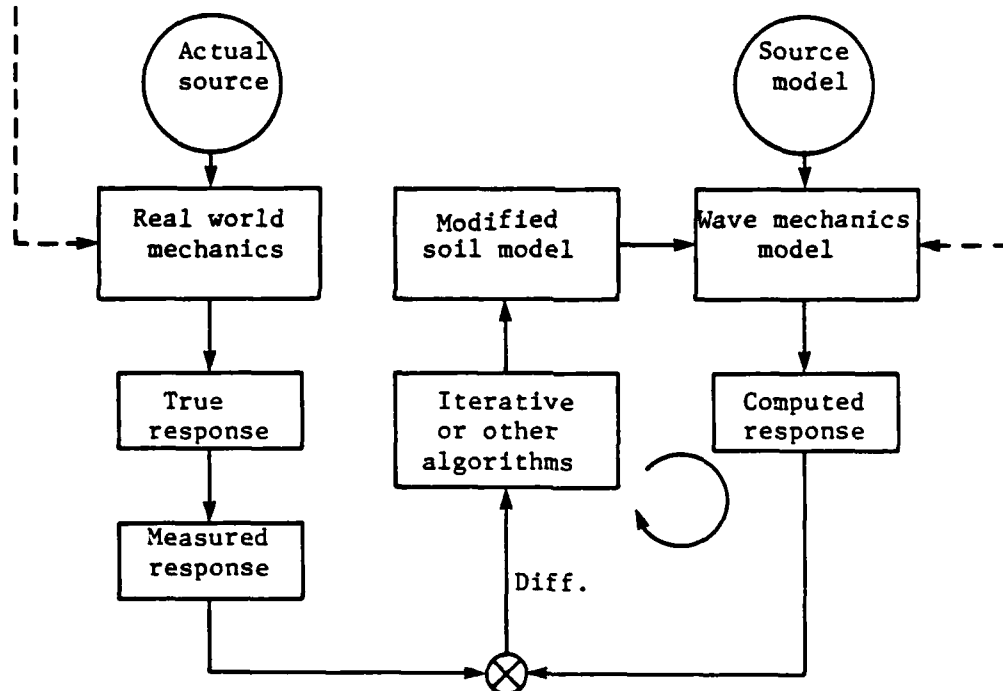
(b) Iterative update by reference to measured test response

Figure 14. Current approach to define a nominal soil model.
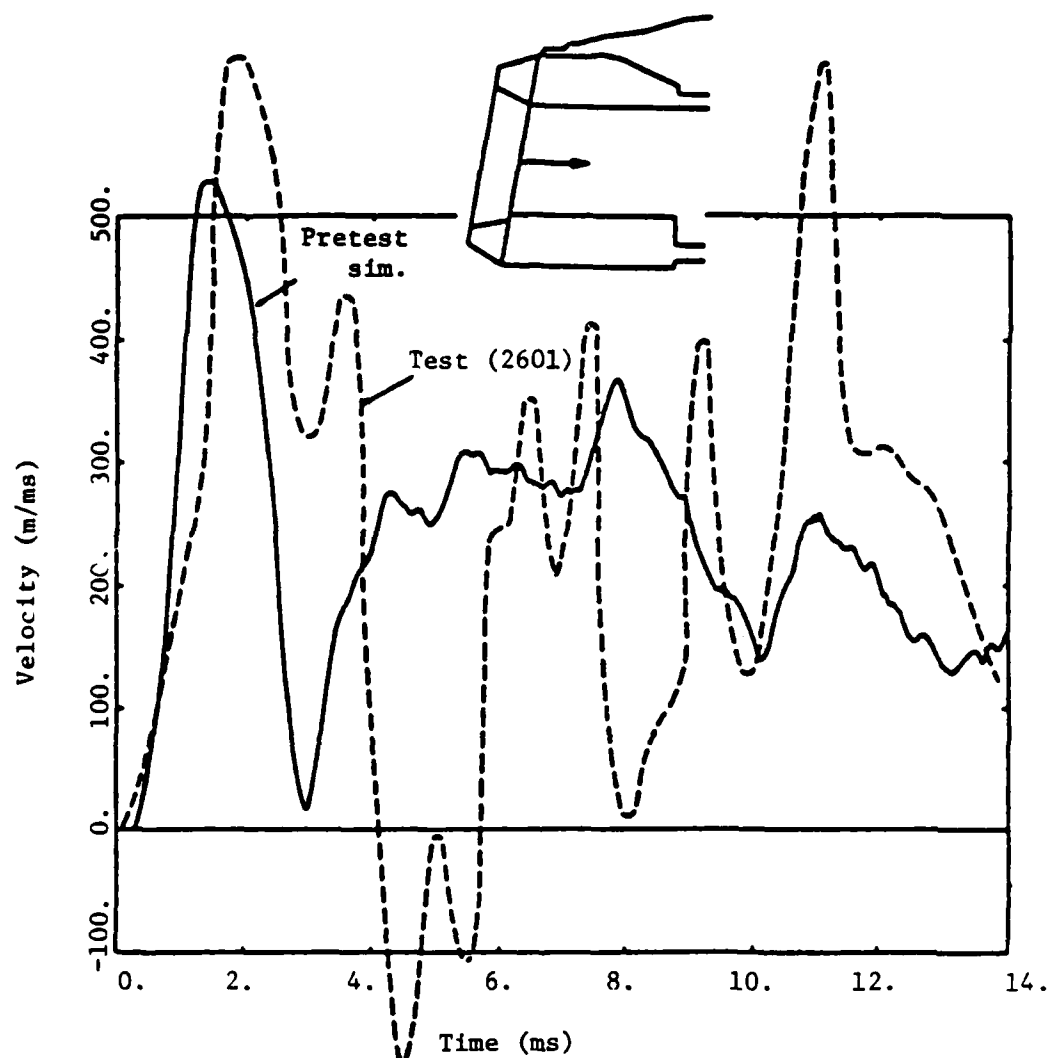
Figure 15. Comparison of measured and simulated structural responses, example 1.
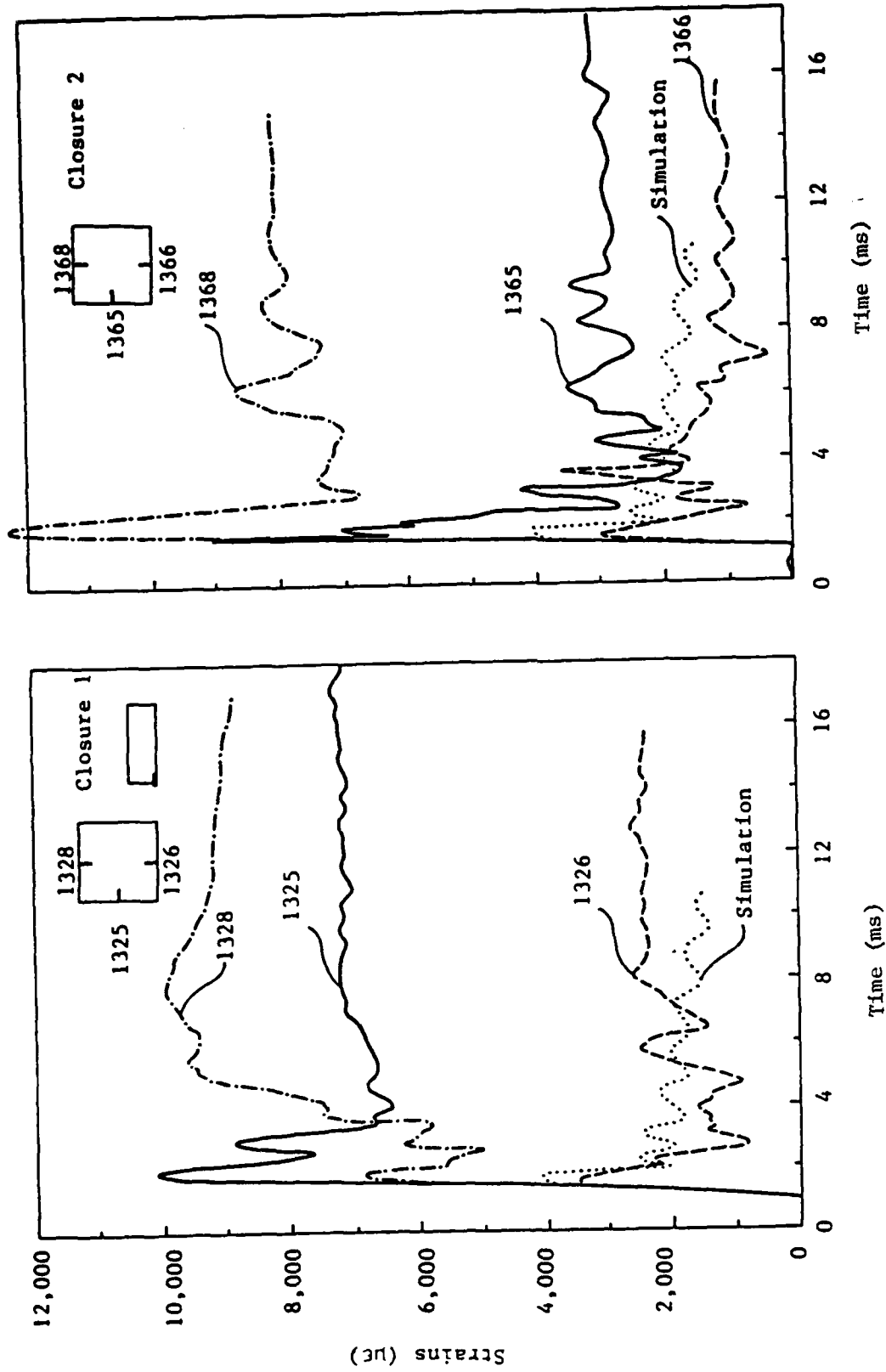
Figure 16. Comparison of measured and simulated structural responses, example 2.

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CLASSIFICATION OF UNCERTAINTIES

From the foregoing discussion, it is obvious that the subject of uncertainty in S/V assessment is both broad and elusive. In this study, the following simple approach is used. Uncertainties are classified either as random or fuzzy, depending on whether they can be appropriately modeled by probabilistic models or not. Hence, fuzzy uncertainties include those resulting from incomplete and imprecise information, subjective judgment, ambiguity and vagueness. This classification is shown in Fig. 17.

The representation in Fig. 17 is intended to be symbolic. There is no crisp demarkation separating the three categories: deterministic, random and fuzzy. A quantity does not become fuzzy, random or deterministic simply by moving a short distance across a boundary. Although there are events which are random or fuzzy, most of the regions overlap. For example, fuzzy uncertainties may include uncertainties which are random in nature but because of the limited data available cannot be adequately characterized as such. When the data base is enlarged, these uncertainties will leave the fuzzy domain to join the random domain. The example on the simple column described earlier is a case in point. On the other hand, some fuzzy uncertainties are nonrandom, irrespective of the size of the data base. Linguistic and expert opinion are two examples of this type. More will be said on this in Section VI.

Hence, fuzzy uncertainty can be associated with one or more of the following characteristics:

(1)   Sample space is not defined;
(2)   Few test specimens are available;
(3)   Average of samples may not be meaningful;
(4)   Linguistic or pictorial data are involved; and
(5)   Judgment and subjectivity are involved.

A more rigorous discussion of random and fuzzy uncertainties requires going into set theories, elements of which will be summarized below. For details the reader is referred to Refs. 9 and 10.

Crisp sets and random uncertainties--Random uncertainties are based on probability theory which is based on classical sets
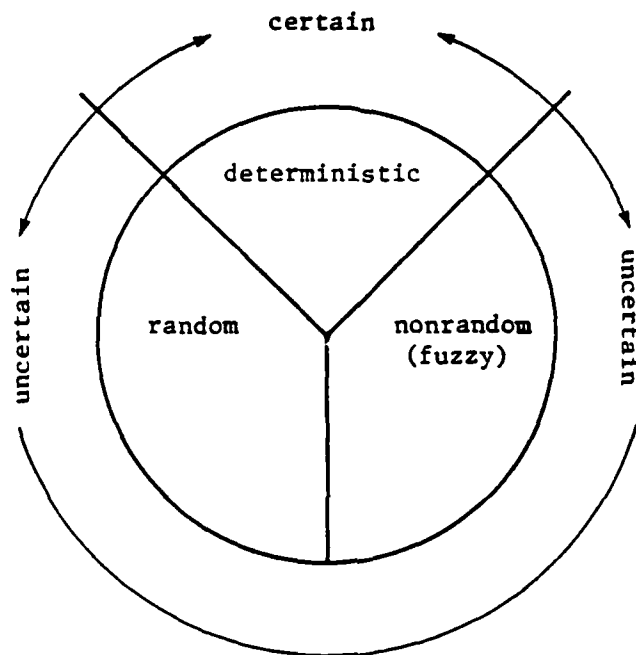
Figure 17.   Classification of uncertainties.

or crisp sets. For example, consider the set of all possible outcomes of an experiment, called universe X. This is denoted by $X = \{ x \mid x \in (0, \infty) \}$ where x is understood to be an element of X. A set of some outcomes in the universe is called a subset of X, and will be denoted by the letters A, B, ... and so on. For example, the subset of concrete strengths between 6 and 9 MPa is the subset $A = \{ x \mid x \in (6,9) \}$. A strength x which belongs to this subset is denoted by $x \in A$, or in terms of the characteristic function $\chi_A(x)$,

$$\chi_A(x) = \begin{cases} 1 & \chi \in A \\ \\ 0 & \chi \notin A \end{cases} \tag{5}$$

In other words, Eq. 5 says that either element x belongs to the subset A, or it does not belong to the subset. In the former case, the membership of x in A (or the belongingness of x in A) is 1. In the latter, the membership is 0. Hence, crisp set is associated with a yes-or-no proposition used to screen its element x, which is also known as binary or two-value logic. The fact that $_A(x)$ can only be either 1 or 0 is sometimes denoted by $\chi_A(x) \in \{0,1\}$.

When the outcome x of an experiment is randomly but uniformly distributed over the universe X, the probability that x belongs to a subset A (called the probability of event A) is

$$p(A) = \sum_{x \in X} \chi_A(x) \frac{1}{X} \tag{6}$$

and, in this case, is simply

$$p(A) = \frac{A}{X} \tag{7}$$

When x is randomly distributed over X with a general distribution $f(x)$, the probability of event A is then

$$p(A) = \sum_{x \in X} \chi_A(x) f(x) \tag{8}$$

Fuzzy sets and fuzzy uncertainties--For events which are not well defined or involve fuzziness (e.g., "concrete strength is around 7 MPa," "structural damage is quite severe"), the concept of crisp sets must be modified. Zadeh (Ref. 9) first proposed generalizing crisp sets to fuzzy sets. In fuzzy sets, the

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transition from membership to non-membership of an element x in a set is gradual, and not abrupt as indicated by the characteristic function in crisp sets. Hence, the natural extension of the characteristic function is the membership function $\mu_A(x)$, where $\mu_A(x)$ represents the grade of membership of x in A where A is now a fuzzy set or event. A membership of 1 corresponds to total belongingness in the set, and 0 denotes the opposite. However, $\mu_A(x)$ is also allowed to have any value in the inclusive range [0,1], and this fact is denoted by $\mu_A(x) \in [0,1]$. This main feature of the membership function, a generalization of the characteristic function, provides the necessary flexibility in fuzzy sets to model fuzzy, noncrisp information. The difference between $\mu_A(x)$ and $\chi_A(x)$ is illustrated in Fig. 18.

Subsequent sections will go into these concepts and theories in greater depths. Suffice it to say here that the approach being proposed is to model random uncertainties using random models, and to model fuzzy uncertainties using fuzzy models. Several major efforts of the study in these two directions will be described in subsequent sections. It is pertinent to ask at this point: Can the two kinds of models and analyses be integrated into an overall assessment framework? Although much more research needs to be performed, the answer is a tentative yes based on some preliminary investigations described in Wong, Ross and Boissonnade (Ref. 5). Some major results are summarized in the following.

INTEGRATING RANDOM AND FUZZY UNCERTAINTIES

It is helpful to recall how random and systematic uncertainties are combined in current methods. Consider the input-response model

$$y = f(x_1, x_2, \ldots \ldots x_N) \qquad (9)$$

where x_i are random parameters, and f is a deterministic model. Modeling or systematic uncertainties are allowed by introducing a multiplicative factor, y_0, so that

$$y = y_0(x_1, x_2, \ldots \ldots x_N) \qquad (10)$$

where y_0 is a random variable designed to take care of inaccuracy in modeling.

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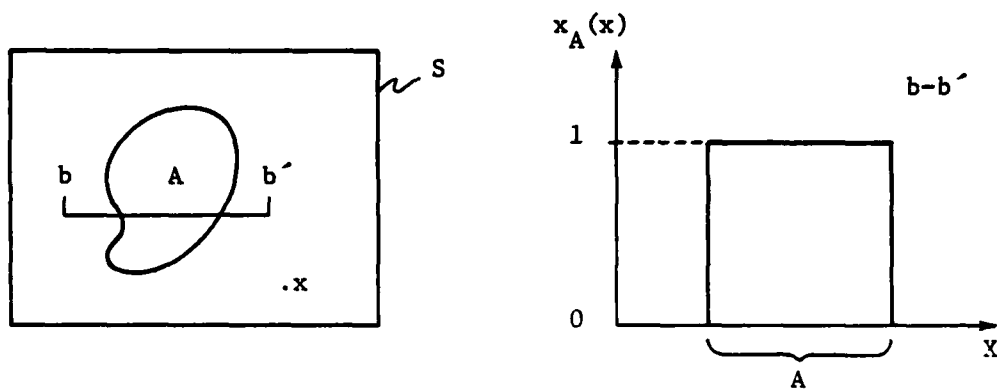
(a) Crisp set A and characteristic function X_A.
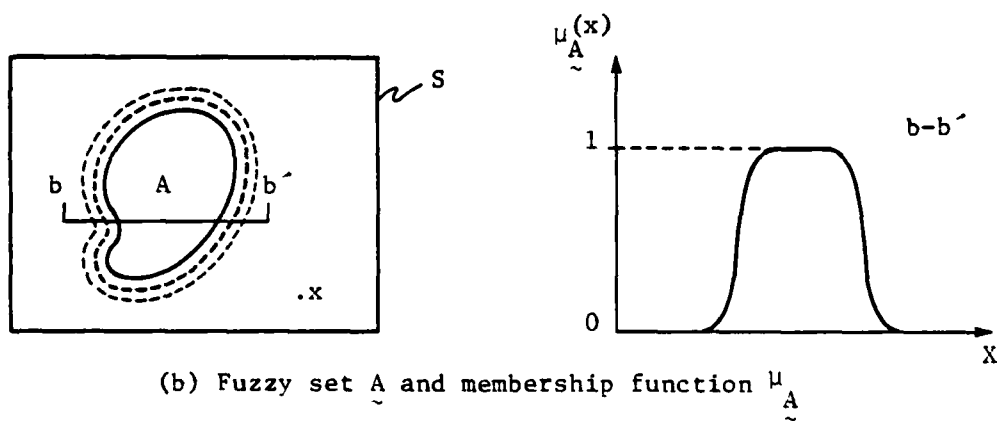
(b) Fuzzy set $\underset{\sim}{A}$ and membership function $\mu_{\underset{\sim}{A}}$

Figure 18.   Graphical comparison of the concepts of crisp and fuzzy sets.

There are now two ways to treat y_o. First, y_o is considered as another parameter in the model, i.e., write Eq. 10 as

$$y = g(x_1, x_2, \ldots\ldots x_N \; ; \; y_o) \qquad (11)$$

This mixes up the random and model uncertainties, with the end result that the system response has a larger variance than when y_o is not included. The result is indicated in Fig. 19a in the context of a fragility curve, i.e., the probability that a component will fail for a particular loading. The consequence in system survivability is tremendous. Such influences were recognized early and the practice is now discontinued.

The second interpretation of y_o is to consider this parameter, although random, to be of a different kind than the random parameters x_1, x_2 ... etc. This leads to prevalent two-level or two-loop methodologies. The consequence can be summarized in Fig. 19b in terms of the fragility curve mentioned previously. The lateral shift or bias of the curve corresponds to the systematic uncertainty y_o. The result is that system survivability is often dominated by y_o (see Goering and Binniger, Ref. 11), since the effect due to random variation tends to average out.

Wong, Ross and Boissonnade (Ref. 5) showed that fuzzy and random uncertainties can be combined in the same manner but there are other flexibilities. The fuzzy models can be established based on available data, even though the data may be scarce or in linguistic form. Figure 20 is an example of the use of fuzzy models to represent fuzzy uncertainties, which in this case concern the degree of damage. One is no longer constrained to the fail/no-fail proposition, nor are implicit assumptions on probabilistic basis of damage necessary. Several other ways to integrate fuzzy descriptions into current probabilistic analysis methods are also described.
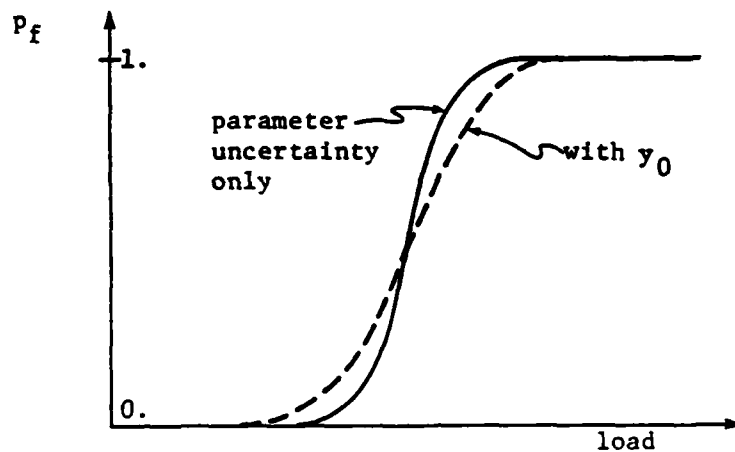
(a) y_0 as one of the random parameters
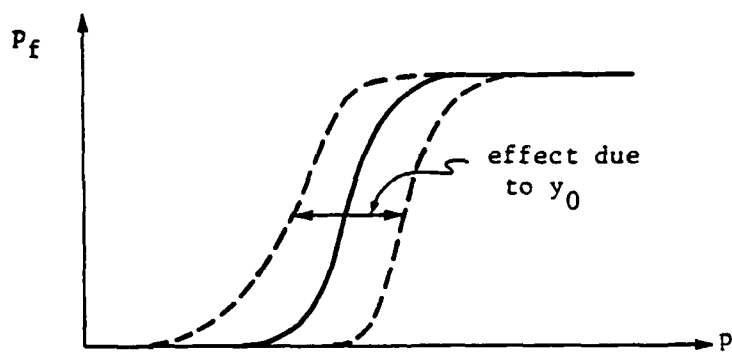
(b) y_0 as a different kind of random parameter

Figure 19. Two possible treatments of modeling (systematic)
uncertainties y_0 in an all-random approach.
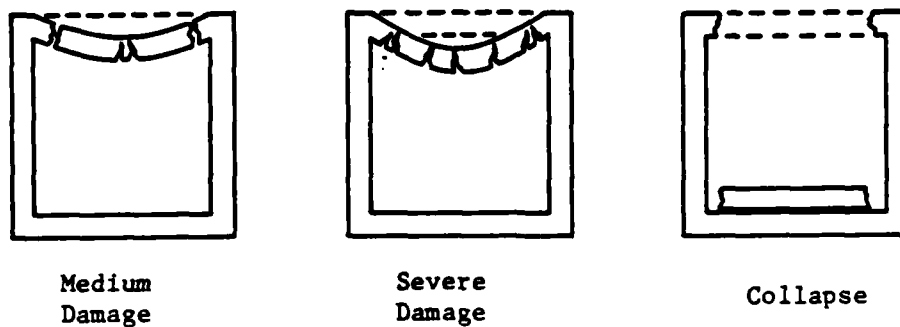
(a) Damage to structure
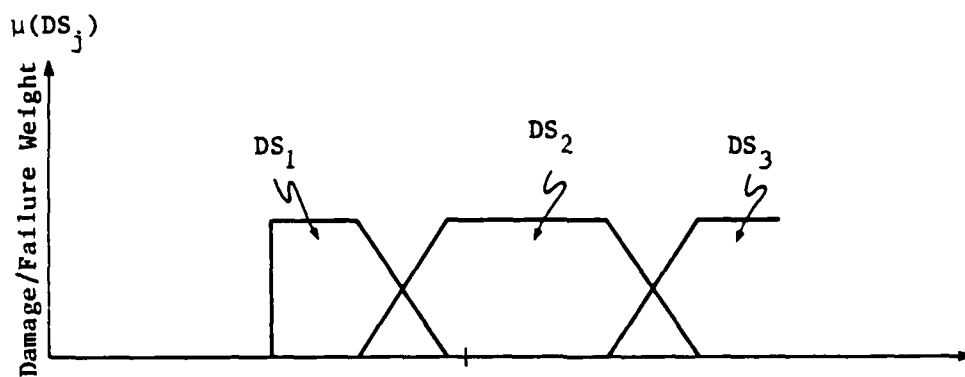
(b) Fuzzy set representation

Figure 20. Nonuniform and overlapping supports for three fuzzy damage states.

IV. EXTENSIONS OF CURRENT CAPABILITIES

Current capabilities in probabilistic S/V analysis can be
represented, without loss of generality, by the flow diagram in
Fig. 21a. The methodology is basically one of propagation of
statistical uncertainties in the input and systems parameters
through the various components of the system model to obtain the
statistical uncertainties in the response. As shown in Fig. 21a,
the major components of the system model include the explosive
effects environment, the load transfer function, the structural
network and component fragilities (see Ref. 2). When reduced to
its essence, the procedure can be represented by the simple
diagram in Fig. 21b.

Extension of this methodology is sought along two major
directions, corresponding to the two classes of uncertainties
identified in the previous section, viz., random and fuzzy.
These extensions are illustrated schematically in Figs. 22 and
23, respectively, and compared with the current approach.

In one research project, current statistical methodology is
extended to the realm of stochastic and random methods. Random
uncertainties are not limited to modeling by statistical param-
eters. More realistic modeling by random and stochastic proces-
ses are sought. This work investigates the use of stochastic and
random differential (integral) equations in probabilistic S/V
analysis, and emphasizes numerical solution techniques such as
stochastic difference and finite element methods. Results are
summarized in Section V.

A second project investigates the application of fuzzy sets
to model fuzzy uncertainties in S/V assessment. In this work,
fuzzy sets representation of uncertainties, including judgment and
opinions, is examined, and the use of fuzzy logic to process fuzzy
uncertainties is considered. The work focuses on the quantifica-
tion of fuzzy information and the interaction between experts and
soft data. Results are summarized in Section VI.

Other related work not described herein include the follow-
ing:
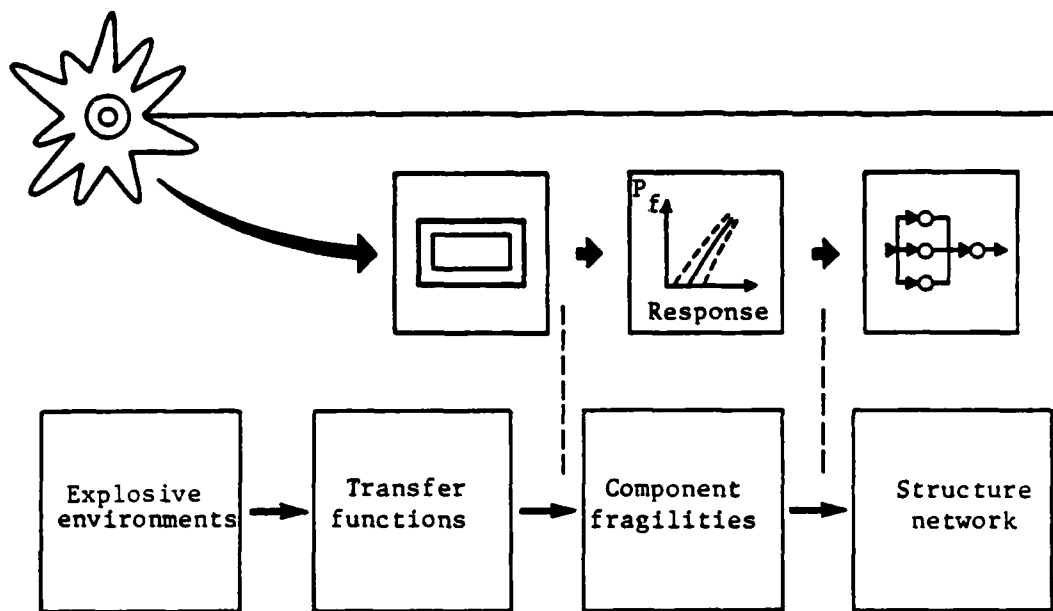
(a) Components of statistical S/V analysis
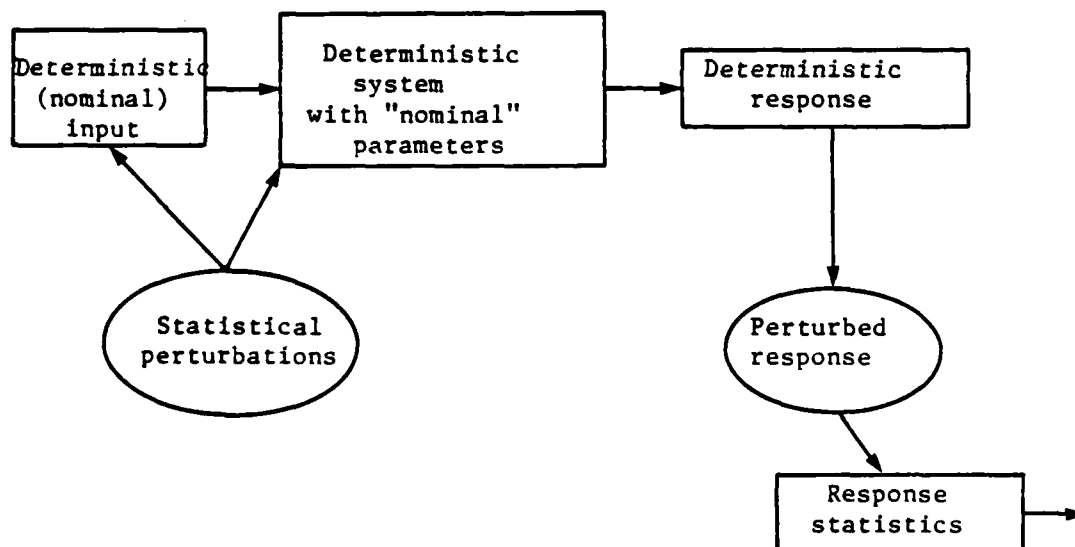
(b) Propagation of statistical uncertainties

Figure 21. Principle of current statistical S/V analysis.

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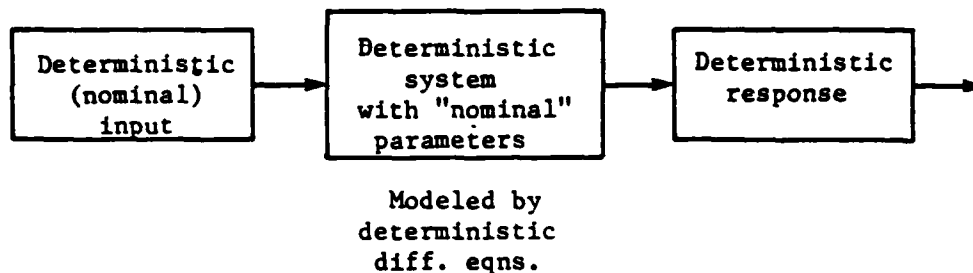
(a) Deterministic approach

(b) Statistical approach based on deterministic model

(c) Stochastic approach

Figure 22. Stochastic/random equation approach to S/V analysis.
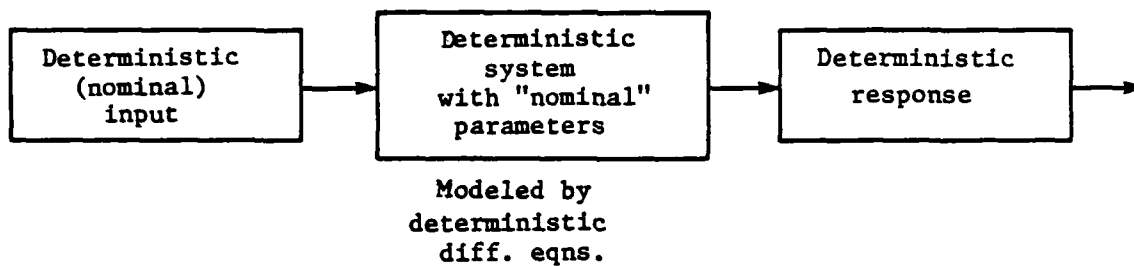
(a) Deterministic approach
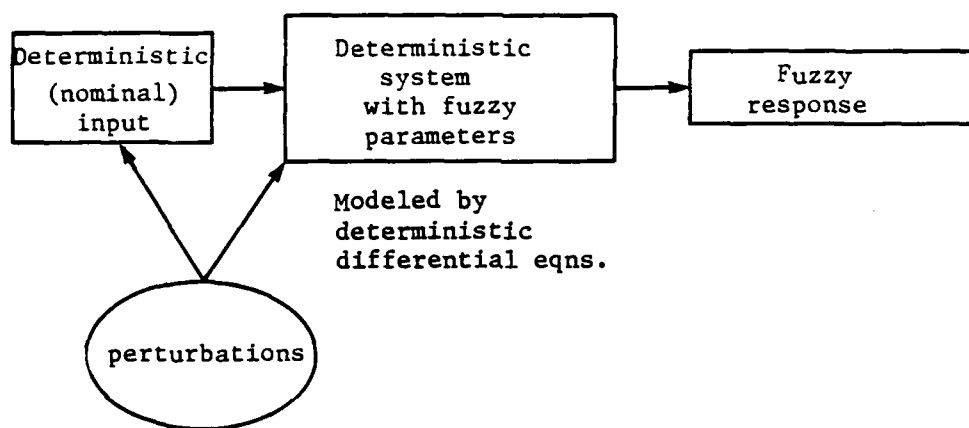
(b) Fuzzy perturbations based on deterministic model
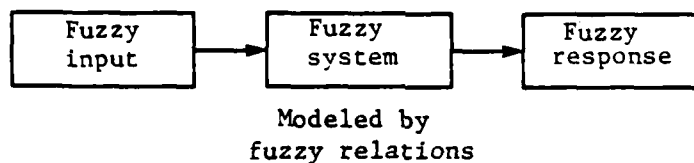
(c) Fuzzy analysis based on fuzzy relations

Figure 23. Fuzzy sets approach to S/V analysis.

(1) **Ways to incorporate more realistic mechanics
models in statistical S/V assessment--**
Sophisticated system models exemplified by the
state-of-the-art dynamic finite element models
used extensively in deterministic S/V analysis
are incorporated into the statistical assessment
framework by the use of transfer function techni-
ques. Two main approaches to transfer function
development are investigated: direct statistical
approximation and engineering approximation. The
former makes use of the response surface methodology
and point estimate methods to develop a statistical
(but simpler) equivalent of the sophisticated system
model. The latter relies on engineering experience
to develop a mechanistic approximation which is then
used in the statistical analysis. Details of this
work can be found in Wong and Richardson (Ref. 4) and
Refs. 12 and 13.

(2) **Stochastic finite element methods to model protective
structures--**When the stochastic effect is separable,
such as in the loading function and initial condition,
discretization methods such as the finite element
method can be applied directly on the continuum equa-
tions (beams, plates, etc.) without any major modifica-
tions. One result is a direct recursive relation
governing the statistical moments of the responses at
consecutive time steps. When the stochastic effect is
non-separable, such as in the coefficient of the equa-
tions or in the form of distributed loading or initial
condition, a bona fide stochastic finite element tech-
nique must be used. One such method uses the Wiener
increment as a basis function in a series expansion.
Details of these works are described in Refs. 14 and
15.

(3) **Comparing random and fuzzy treatments of uncertain-
ties in structural models--**The effect of parameter
uncertainty on structural response is described, first
using the stochastic approach and then the fuzzy set
approach. The stochastic model of parameter uncertainty
leads to the paradox that the most probable response

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differs from the response of the most probable structure, and this difference can be significant. Fuzzy set models of parameter uncertainty lead to fuzzy response, which is judged more consistent with intuition. This work is described in Ref. 16.

(4) **Development of a knowledge- and rule-based assessment system**--A knowledge-based assessment system processes knowledge and not just numbers. The knowledge is usually expressed in the form of rules which can be manipulated at the symbolic level. A different approach to the formulation of the rules is considered in this study. The approach is based on fuzzy set representation of knowledge and inferences, and rules are embodied in fuzzy relations. This approach allows a natural way to combine rules which contain uncertainties and may also be in conflict with one another. Details are given in Ref. 17.

V. RANDOM UNCERTAINTIES AND RANDOM EQUATIONS

The organization of this section is as follows. The different types of random equations are first described. Emphasis is on the type which involves random processes (including random constants and random functions as special cases) in the coefficients of the equation. Distinction is also made between random and stochastic types--the latter involving the Wiener process or its formal derivative, the white noise. Methods of solution to random/stochastic differential equations through their difference or algebraic counterparts are the focus of the study and will be discussed in this section. It is assumed that the reader has some knowledge of probability theory and ordinary differential equations. A brief introduction to random equations and stochastic integrals is included to make the following discussion relatively self-contained. More information is available in the cited references.

TYPES OF RANDOM DIFFERENTIAL EQUATIONS

A random differential equation is an equation which satisfies at least one of the following conditions:

(1)   The initial or boundary condition is random;
(2)   The inhomogeneous term (forcing function) is random; and
(3)   At least one of the coefficients is random.

Here, random means a random variable, a random function or a random process, whenever applicable. A random function is defined to be a random process which can be defined by a finite number of random variables. The most common classification of random equations is done according to the order (1), (2) and (3), which also corresponds to a hierarchy of increasing mathematical difficulty.

Homogeneous equations with random initial conditions--This is the simplest type of random differential equations, and the solution can be readily obtained for both linear and nonlinear systems. The generic equation has the form

$$x'(t) = f(t) \, x(t) \quad , \quad x(t_0) = x_0 \qquad (12)$$

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in the linear case, and

$$x'(t) = f(x(t),t) \quad , \quad x(t_0) = x_0 \qquad (13)$$

in the nonlinear case, where x_0 is a random vector.

The solution $x(t)$ obviously depends on x_0 in addition to t, and owes its random character to x_0. The simplicity comes from the fact that the governing equation can be considered a transformation between $x(t_0)$ and $x(t)$ at any time t. Furthermore, using deterministic theory, this transformation can be obtained by solving the governing equation for a deterministic x_0 (see Soong, Ref. 18).

Equations with random excitations--This type of random differential equation represents the next level in difficulty, and a majority of the research efforts in the *past twenty years is* devoted to this group under the name of random vibration. The generic form is

$$x'(t) = f(x(t),t) + y(t) \quad , \quad x(t_0) = x_0 \qquad (14)$$

where $y(t)$ is a random process and the initial condition x_0 can be random or deterministic.

For the linear case where $f(x,t) = f(t)x(t)$, the solution is

$$x(t) = \Phi(t,t_0) \, x_0 + \int_{t_0}^{t} \Phi(t,s)y(s)ds \qquad (15)$$

where   is the principal matrix associated with $f(t)$. Many results in random vibration theory are based on this solution (see Elishakoff, Ref. 19).

Equations with random coefficients--This represents the most complicated class of problems, and has the generic form

$$x'(t) + a(t)x(t) = y(t) \quad , \quad x(t_0) = x_0 \qquad (16)$$

in the linear case, where $a(t)$ and $y(t)$ are random processes, and x_0 is random or deterministic. The study of systems and structures having imprecise parameter values or inherent imperfections leads to differential equations of this type. For S/V

applications, the governing equation can also be nonlinear (in-
elastic), and the difficulty of the problem is further increased.

Note that when the coefficients are not random processes,
but random constants or functions of random constants, a much
simpler solution approach is available. An equation of the type

$$x'(t) = f(x(t), a, t) \quad , \quad x(t_0) = x_0 \tag{17}$$

where the random vector a is constant in t can be rearranged to
become

$$z'(t) = k(z(t), t), \quad z(t_0) = z_0 \tag{18}$$

where z(t) is the augmented state vector

$$z(t) = \left\{ \begin{matrix} x(t) \\ \\ a \end{matrix} \right\} \tag{19}$$

In terms of z(t), Eq. 18 describes a vector differential equation
where randomness enters only through the initial condition.
Hence, methods described earlier for this class of equation are
applicable to the augmented equations (see also Ref.18).

ANALYTICAL DIFFICULTIES

Of the three types of random equations, the third type is
the most difficult to solve but includes the problems of present
interest, i.e., transient, inelastic response of structures with
random properties and subjected to random loading. To appreciate
the difficulty associated with equations with random coeffi-
cients, consider the simplest first-order linear differential
equation of the form expressed in Eq. 16. With certain assump-
tions on the properties of a(t), the solution can be written by
direct quadrature formally as

$$x(t) = x_0 \exp\left[- \int_{t_0}^{t} a(s)ds \right] + \int_{t_0}^{t} y(u) \exp\left[- \int_{u}^{t} a(s)ds \right] du \tag{20}$$

It is seen that the coefficient process a(t) enters into the
solution in a complex way. In particular, the dependence of x(t)
on a(t) is nonlinear, despite the fact that the governing

equation is linear. This nonlinear relationship creates much of
the difficulty encountered in equations of this type. The
knowledge of the density functions of $a(t)$ is in general neces-
sary for determining even the simplest moments of the solution
process. The joint probabilistic behavior of $a(t)$, $y(t)$, and x_0
is also required to solve the general sample path, $x(t)$.

Perhaps the largest group of problems of this type which has
been approached with some success is that involving the Wiener
process or its formal derivative, white noise. The governing
equations are referred to as equations of the Ito type, and a
number of approaches can be used. It is simple to establish the
Fokker-Planck equation or the moment equations for the Ito equa-
tion. However, the solutions of the Fokker-Planck equations are
difficult to obtain except for trivial cases. For engineering
problems, most success comes from using the associated moment
equations.

Two important points need to be made. First, the quadrature
solution given by Eq. 20 may not be valid when $a(t)$ is not well-
behaved. This is the case when $a(t)$ is a white noise process,
and the original equation must be approached using Ito calculus,
i.e., as a stochastic equation. Hence, it is important to dis-
tinguish between the general random differential equations and
the special group called stochastic differential equations.
Second, stochastic differential equations have special properties
which, on the one hand, make analytic solutions possible, but on
the other hand create additional complications when numerical
solutions are sought. These points will be discussed further
presently.

The classification outlined above is summarized schemati-
cally in Fig. 24. More details are given in a literature survey
documented in Refs. 20 and 21.

NUMERICAL TECHNIQUES

There are several ways by which a general solution to a
random differential equation can be approached. The equation can
be solved in closed form, although this approach is very limited;
only a few classical examples are known to exist and they are
summarized in Ref. 21. The equation can also be reformulated
into a random integral equation, approximate solutions to which
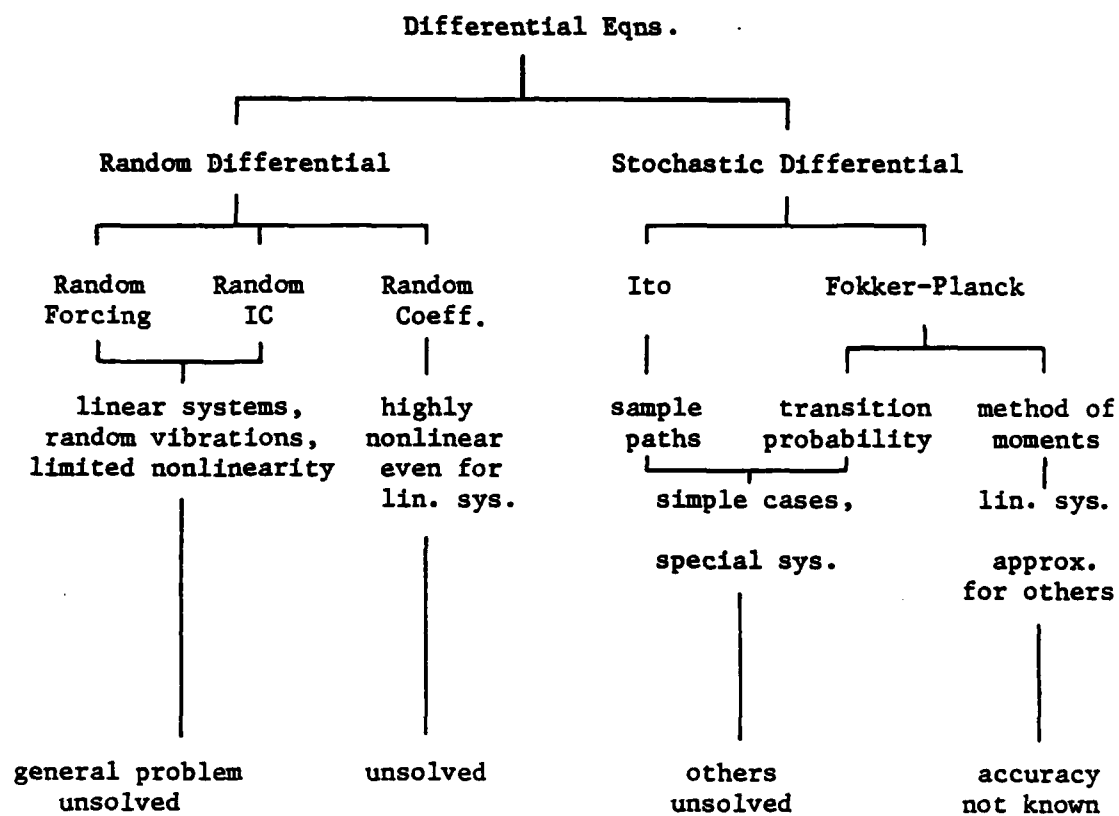
Figure 24. Basic types of random equations.

can be approached using series expansion or discrete quadrature.
The former results in a recursive algorithm and a comparative
study of available techniques of this kind is described in Ref.
22. The latter results in a set of simultaneous algebraic equa-
tions and this approach is championed mainly by Bharucha-Reid
(Ref. 23). In either case, approximate numerical solution is
necessary.

Emphasis of this study is on numerical techniques applied
directly to the differential equations. It is argued that, if
the computer is to be used, it may as well be applied directly.
The numerical approach is deemed appropriate also because of the
difficulty in solving the equations otherwise, and equations
which correspond to ^/V appliations of interest will be even more
complicated. The end result of the numerical approach is invari-
ably a random difference equation which is similar in form, and
yet may be quite different from the result of a finite element or
finite difference formulation in the deterministic counterpart.

Distinction is made in the following discussion between
random difference equations and stochastic difference equations.
In particular, we are interested in the relationship between the
mean of the solution from the random and stochastic equations and
the corresponding solution obtained from its deterministic coun-
terpart, i.e., when the loading and system parameters are set at
their mean values. Of interest also are the statistics of the
solution, e.g., variance and correlation, and how they are re-
lated to the statistics obtained from a conventional statistical
study based on the deterministic formulation. The latter is, of
course, the state of the art in S/V analysis.

FINITE DIFFERENCE SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

This subsection describes the use of finite difference tech-
niques to solve stochastic differential equations, i.e., equa-
tions with the formal form

$$dx_t = a(t, x_t)\, dt + b(t, x_t)\, dw_t \qquad (21)$$

where w_t is the Wiener process and dw_t is its formal derivative,
or the white noise process. In this approach, the derivative
operator $d(.)$ in Eq. 21 is formally replaced by an equivalent

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difference operator (.). Equation 21 then becomes a stochastic difference equation

$$\Delta x_t = a(t,x_t)\Delta t + b(t,x_t)\Delta w_t \qquad (22)$$

However, because of the special properties of the Wiener process, such implementation of the difference approximation is not straightforward. There are several differences between this application and those, say, of deterministic differential equations. For example, the difference equivalent, w_t, of the white noise process, dw_t, is a difficult subject which has occupied many prominent mathematicians.

A mathematically more rigorous representation of the stochastic equation is in the form of an integral equation

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} a(s,x_s)ds + \int_{t_1}^{t_2} b(s,x_s)dw_s \qquad (23)$$

where the first integral is an ordinary integral and the second integral is a stochastic integral. A difference approximation to Eq. 23 is obtained by letting $t_2 = t_1 + t$, where t is the time increment. This gives,

$$x(t_1 + \Delta t) = x(t_1) + \int_{t_1}^{t_1 + \Delta t} a(s,x_s)ds + \int_{t_1}^{t_1 + \Delta t} b(s,x_s)dw_s \qquad (24)$$

Discretization techniques are then used to express both the ordinary integral and the stochastic integral as finite sums, usually with only one term. Notationally, Eqs. 21 and 22 are more expedient than Eqs. 23 and 24, and will be used in subsequent discussion. The true meaning of these equations should not be forgotten, however.

Four important topics in assessing the feasibility of using the difference techniques in stochastic applications are:

(1) The Wiener increment sequence Δw_t that can be generated to drive the difference equation, Eq. 22;

(2) The different Δ algorithms that can be used and their convergence characteristics;

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(3) The solutions to which these algorithms converge; and

(4) The response statistics that can be computed from the difference solutions.

Results of these studies are summarized below, and details can be found in Ref. 24. Note that the usual error analyses of difference methods in connection with deterministic differential equations do not apply to stochastic equations (such as Eq. 21). The reason is that any sample of the white noise process dw_t is required to be everywhere unbounded, discontinuous, and non-differentiable.

<u>Wiener increments</u>--The Wiener increment Δw_t denotes the difference between the values of the Wiener process at two instances of time, i.e.,

$$\Delta w_t = w(t + \Delta t) - w(t) \qquad (25)$$

and is known to have the following properties:

$$<\Delta w_t> = 0, \quad <\Delta w_t \Delta w_t> = \sigma^2 \Delta t \qquad (26)$$

Similar properties can be written for a vector Wiener increment and the results described in the following can be extended readily to a vector stochastic differential equation. The parameter σ is called the intensity of the Wiener increment process. When $\sigma = 1$, the process is called a normal or standard Wiener process, and this will be assumed in subsequent discussions.

The first task in numerical solution of stochastic differential equations is to generate a sequence of pseudorandom numbers on the computer which can be used to approximate the Wiener increment process. Franklin (Ref. 25) is apparently the first to consider this problem. His analysis shows that if g_r, $r = 1, 2, \ldots$ denotes a sequence of pseudorandom Gaussian numbers with zero mean and unit variance, i.e.,

$$g_r = N(0,1) , \quad r = 1,2,\ldots \qquad (27)$$

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in the usual notations, then the Wiener increments can be simulated by

$$\Delta w_r = \sqrt{\Delta t}\; g_r, \quad r = 1, 2, \ldots \ldots \quad (28)$$

where Δt is the time step used in the difference approximation.

It is important to note that the Wiener increments given by Eq. 28 depend on Δt and, hence, the power of the increment sequence depends on the time step size. In other words, when the time step is decreased, as in a convergence study to be described presently, the power of the excitation changes. This behavior is undesirable, and what is needed is a scheme by which the power of the Wiener increment sequence can be made to remain constant as the time step is varied. McCallum (Ref. 26) provides such a procedure, and details are given in Ref. 24.

Difference algorithms--Formally, all difference algorithms in use with deterministic differential equations can be used with stochastic differential equations. However, these algorithms all lead to different solutions for the same stochastic differential equation--a phenomenon unlike any other application of the difference method. The cause of such phenomenon can be traced to the properties of the stochastic integral, i.e.,

$$\int_a^b g(x,t)\,dw_t \quad (29)$$

where w_t is the Wiener process, and x is the response of the stochastic equation. More will be said about this in subsequent paragraphs. Briefly, the Euler algorithm will lead to the Ito solution, the Runge-Kutta of order 2 or Heun algorithm will lead to the Stratonovich solution, and the predictor algorithms will lead to the McShane solution. The numerical solution using predictor-corrector algorithms does not correspond to any known stochastic solution.

Hence, from a practical point of view, only the Euler or Heun are viable techniques, resulting in either the Ito or Stratonovich solution. They have different convergence efficiencies, however, and careful planning is needed to obtain the most economical difference procedure which leads to a known interpretation of the stochastic solution.

Convergence--Convergence in this study is defined in probabilistic terms. The most common measure is convergence in probability of the order of t, or $O_p(t)$, which means that

$$O_p(\Delta t^{\alpha}) \longrightarrow \lim_{\Delta t \to 0} \Delta t^{-\alpha} \times \text{Prob}(|\text{error}| > \varepsilon) = 0 \quad (30)$$

where Prob $(|\text{error}| > \varepsilon)$ denotes the probability that the difference between the actual and the approximate solutions is larger than an arbitrary small positive quantity ε.

There have been many theoretical studies of the convergence rate of stochastic difference algorithms. The study by Franklin (Ref. 25) appears to be the first, and the most recent appears to be Rümelin (Ref. 27). Because conventional convergence analysis techniques cannot be used here due to the unboundedness and non-differentiability of the white noise process, most of the convergence proofs given in the literature are very complex, and invoke knowledge of stochastic integrals. Details can be found in Ref. 27. The results can be summarized as follows: The Euler algorithm is $O_p(\Delta t^{1/2})$, and the Heun algorithm is $O_p(\Delta t^{3/2})$. They converge to different solutions, however.

Higher convergence algorithms can be realized by multiple integration of the stochastic integrals (Eq. 29), in contrast to the single integration used in the Euler and Heun algorithms. This is shown by Rao et al. (Ref. 28) and Rumelin (Ref. 27). The use of more multiple stochastic integrals gives more information about the Wiener process component of the algorithms. However, our simulation studies have shown that the higher convergence rate claimed by some of these theoretical studies is not there. For example, the study by Rao et al. was repeated, but the improved convergence cited by the authors did not materialize in our simulation results.

Accuracy--The stochastic integral of Eq. 29 has several definitions, of which three are very well-known: Ito, Stratonovich and McShane. The definitions are based on defining the integral as the limit of a finite sum which involves terms of the integrand evaluated at discrete points within the interval of integration and, of course, the Wiener increments. The difference is in the points at which the integrands are evaluated and how they are used.

Suppose the interval [a,b] is divided into n time segments
of Δt each. The Ito, Stratonovich and McShane definitions of the
stochastic integral are:

Ito

$$\int_a^b g(t)\ dw_t = \lim_{\Delta t \to 0} \sum_{r=0}^{n-1} g(t_r)\ [w(t_{r+1}) - w(t_r)] \quad (31)$$

Stratonovich

$$\int_a^b g(t) dw_t = \lim_{\Delta t \to 0} \sum_{n=0}^{n-1} g[\tfrac{1}{2}(t_r + t_{r+1})]\ [w(t_{r+1}) - w(t_r)]\ (32)$$

McShane

$$\int_a^b g(t) dw_t = \lim_{(t_{r+1} - t_s) \to 0} \sum_{r=0}^{n-1} g(t_s)\ [w(t_{r+1}) - w(t_r)]\ (33)$$

$$t_s \le t_r$$

Hence, it is clear that the Ito definition uses the value of the
integrand at the beginning of each time step, the Stratonovich
definition uses the mid-point of the time step, and the McShane
definition uses points prior to the time step of interest. This
is illustrated in Fig. 25.

Note the similarity between the forms displayed in Eqs. 31-
33 and the Euler and Heun difference approximations given below:

Euler

$$x_{i+1} = x_i + a(t_i, x_i)\ \Delta t + b(t_i, x_i) \Delta w_i \qquad (34)$$

Heun

$$x_{i+1} = x_i + \frac{1}{2}\ \left[a(t_i, x_i) + a(t_{i+1},\ \tilde{x}_{i+1}) \right] \Delta t$$

$$+ \frac{1}{2}\ \left[b(t_i, x_i) + b(t_{i+1},\ \tilde{x}_{i+1}) \right] \Delta w_i \qquad (35)$$

$$\tilde{x}_{i+1} = x_i + a(t_i, x_i)\ \Delta t + b(t_i, x_i)\ \Delta w_i$$

$$\int_a^b g(t)dw_t = \lim_{\substack{\Delta t \to 0}} \sum_{r=0}^{n-1} g(t_r) [w(t_{r+1}) - w(t_r)]$$

Ito Definition

$$\int_a^b g(t)dw_t = \lim_{\substack{\Delta t \to 0}} \sum_{r=0}^{n-1} g[0.5(t_r + t_{r+1})][w(t_{r+1}) - w(t_r)]$$
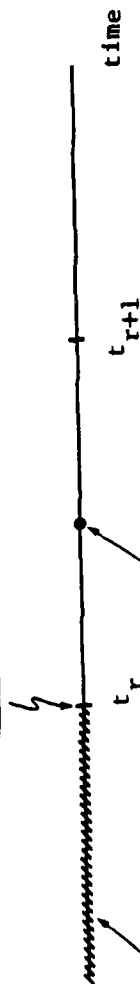
Stratonovich Definition

$$\int_a^b g(t)dw_t = \lim_{\substack{(t_{r+1} - t_s) \to 0}} \sum_{r=0}^{n-1} g(t_s)[w(t_{r+1}) - w(t_r)]$$
$$t_s < t_r$$

McShane Definition

Figure 25. Three definitions of the stochastic integral $\int_a^b g(t)dw_t$.

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It is easy to note the correspondence between the Ito form and the Euler algorithm since both make use of information at the beginning of the time step. Similarly, the correspondence between the Stratonovich form and the Heun algorithm is also obvious--both make use of information at the mid-point of the time step. Finally, the McShane form and predictor methods use past information of the current time step.

With this in mind, it is not surprising that when the Euler algorithm is used to solve the stochastic differential equation (Eq. 21), one obtains the Ito solution; when the Heun algorithm is used, one obtains the Stratonovich solution. Predictor schemes will give the McShane solution. Extending this correspondence further, predictor-corrector algorithms will give a solution which will not be in the Ito-Stratonovich-McShane group. It will be something in between. It is also clear why different difference algorithms give different solutions to the same stochastic differential equation. These findings have also been confirmed by a numerical example described in the following.

Numerical example--The simple equation

$$dx_t = x_t dw_t \qquad (36)$$

is chosen for consideration mainly because its exact solution (Ito, Stratonovich, McShane) is known. In particular, the Ito solution is

$$x_t = \exp\ (w_t - 0.5t) \qquad (37)$$

Note the solution is not $x_t = \exp(w_t)$ as ordinary calculus indicates. This is, of course, a well-known feature of the Ito calculus.

The Euler approximation to the example equation is then

$$x(t_{r+1}) = x(t_r) \left[w(t_{r+1}) - w(t_r) \right] + x(t_r) \qquad (38)$$

whereas the Heun approximation is

$$x(t_{r+1}) = \frac{1}{2}\left[x(t_r) + \tilde{x}(t_{r+1})\right]\left[w(t_{r+1}) - w(t_r)\right] + x(t_r)$$

(39)

$$\tilde{x}_{r+1} = x(t_r)\left[w(t_{r+1}) - w(t_r)\right] + x(t_r)$$

These approximations are computed for different time step sizes and compared with analytical solutions of the Ito and Stratonovich definitions. The results are plotted in Fig. 26 where the ordinate corresponds to the stochastic response at the end of the unit time interval and the abscissa corresponds to the number of time steps in the interval. Similar comparisons for responses at other times and other samples of the Wiener sequence are obtained but will not be shown.

Figure 26 shows very clearly the better convergence properties of the Heun approximation compared with the Euler approximation. It also shows that the Heun approximation converges to the Stratonovich solution, viz., exp(w(1)), where w(1) is the value of the reference Wiener sequence at t = 1. The Euler approximation, on the other hand, converges to the Ito solution, viz., exp(w(1)-0.5).

Since the Heun algorithm has better convergence properties than the Euler algorithm, it is the preferred approximation technique. However, the Heun solution converges not to the Ito solution, but to the Stratonovich solution. A dilemma exists: The more efficient algorithm gives the wrong answer, assuming that the Ito solution is sought. The Ito solution is desired because it is a Markov process which has some very desirable properties. The Stratonovich solution is not Markov. However, to get the Ito solution using the Euler algorithm, convergence is slow.

A resolution of this dilemma is found in a well-known result in stochastic integrals. The Ito solution of the general stochastic differential equation (Eq. 21) underlines(coincides) with the Stratonovich solution of the following equation (which is similar to Eq. 21 and yet different [see Wong and Zakai, Ref. 29]),

$$dx_t = \left[a(t,x_t) - \frac{1}{2} b(t,x_t) \frac{\partial b(t,x_t)}{\partial x}\right] + b(t,x_t)dw_t$$

(40)

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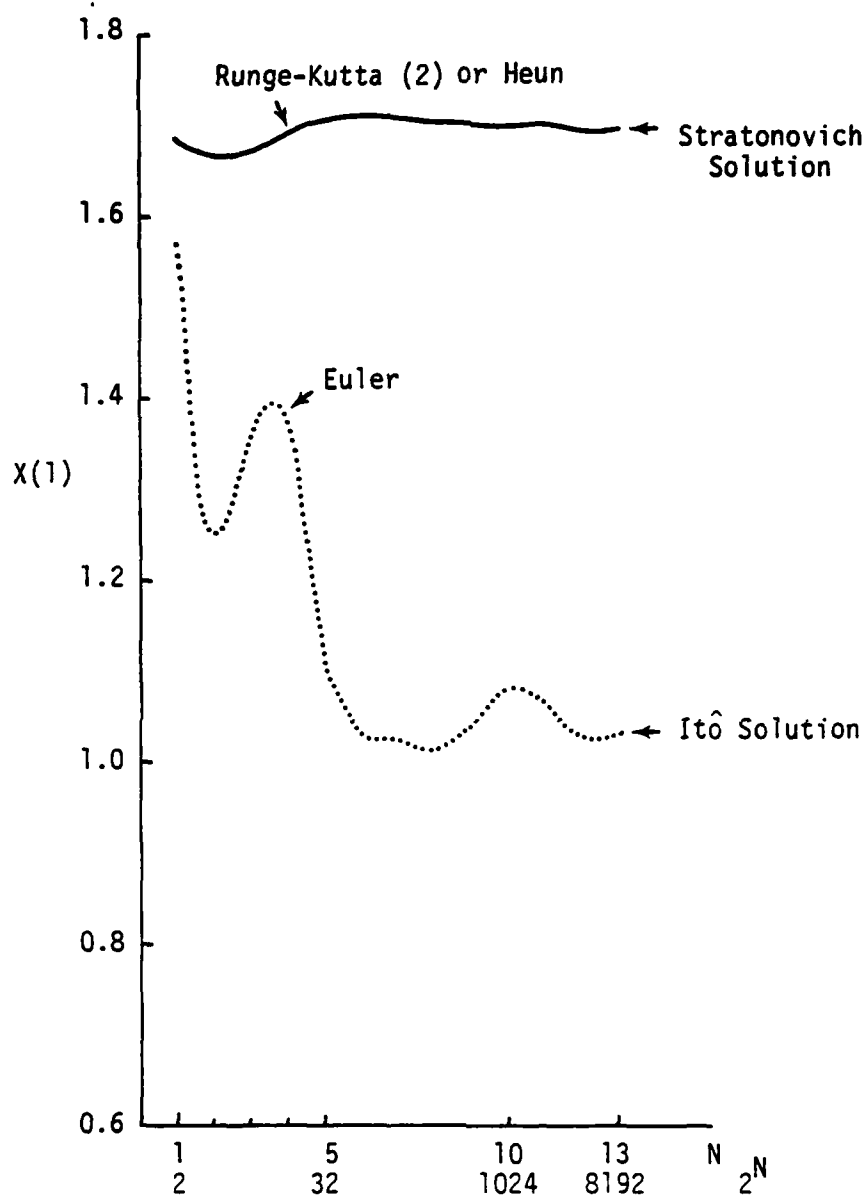
Figure 26. Comparison of Euler and Heun approximations and their convergence properties.

or the alternate equation (see McShane, Ref. 30),

$$dx_t = a(t,x_t) \, dt - \frac{1}{2} b(t,x_t) \, \frac{\partial b(t,x_t)}{\partial x} \, (dw_t^2) + b(t,x_t)dw_t \quad (41)$$

Hence, one can replace the Ito equation by its Stratonovich equivalent, in the sense of Wong-Zakai or McShane, and obtain an approximation solution to the equivalent equation using the Heun algorithm. In this manner, a faster converging approximation is obtained which also converges to the Ito solution. Details are described in Ref. 24.

It should be added that the somewhat nonuniform convergence behavior observed in the results of Fig. 26 is due only partly to the difference algorithms. Numerics of the random number generators is a major contributor. The pseudorandom number generator (IMSL routines) is far from perfect, with the result that the random numbers generated are not truly Gaussian. A detailed discussion on this point is given in Ref. 24, but note that this defect is present in all computer methods. Figure 27 shows the mean and standard deviation of the $N(0,1)$ samples (i.e., normal process with mean zero and standard deviation of 1) as a function of the sample size 2^N. The figure shows that a sample size of 1000, or 2^{10}, is required to reproduce the desired statistics.

Response statistics--The finite difference solutions provide sample responses to the stochastic differential equations. For engineering applications, statistics of the response such as mean and autocorrelation functions are of interest. In principle, it is not difficult to compute these statistical properties of the response process knowing the sample responses. A sufficiently large collection of samples is generated on the computer using the method described in the previous subsections, and the sample statistics can then be computed. In practice, it is found that the sample size required is quite large, and the computational resources are often stretched for problems of interest.

To illustrate this point, return to the example considered. Five thousand sample responses are computed. The first sample moment (mean) and the second sample moment corresponding to this sample population are computed, as well as the moments for subsets of the size 100, 500, 1000 and 2000. They are then compared with the exact moments which can be computed readily for this
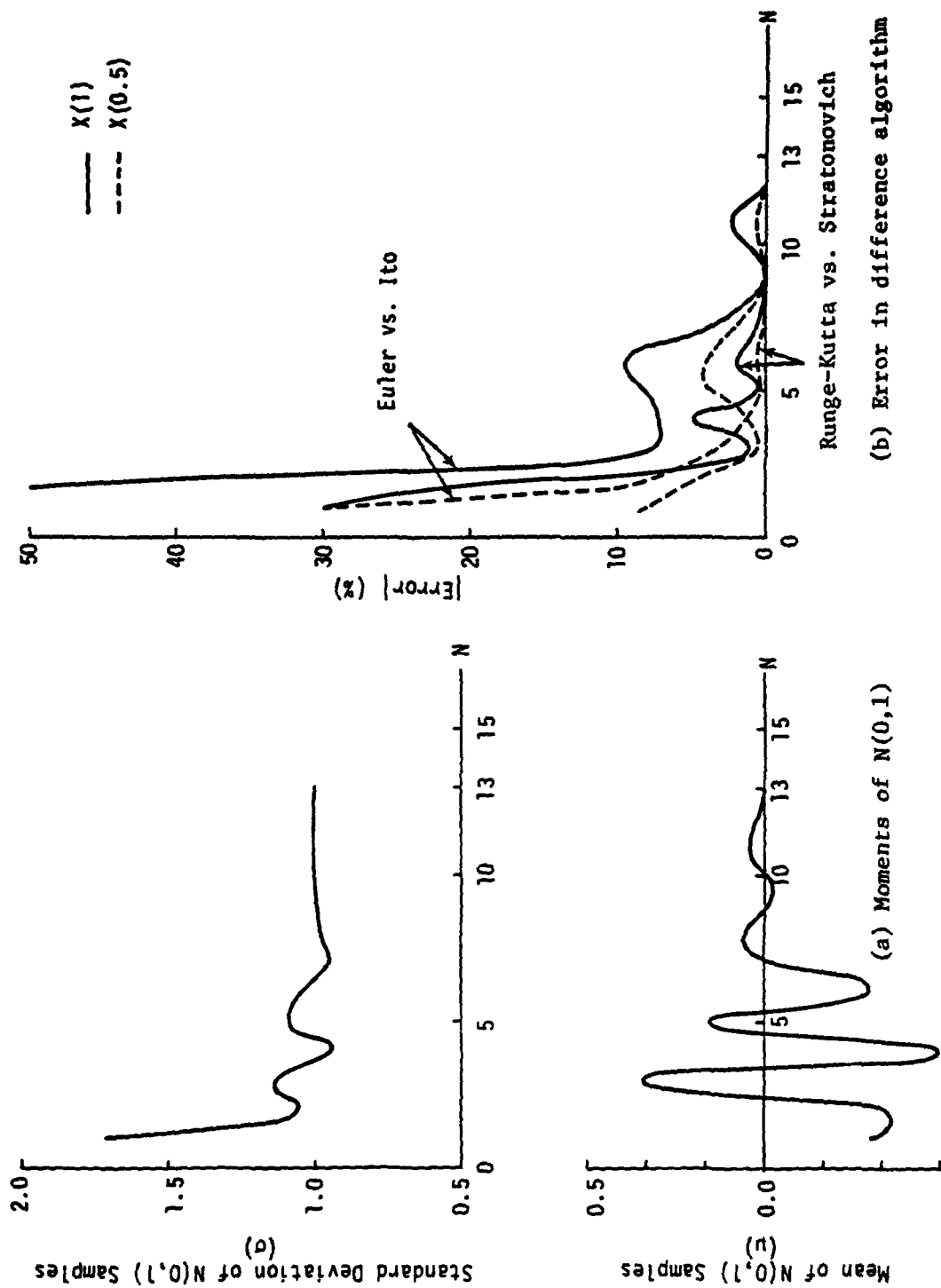
Figure 27. Error and statistics of noise sample as function of timestep size.

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simple example (see Ref. 24). The error defined as the absolute value of the difference between the approximate and exact moments expressed as a percentage of the exact moment is plotted in Fig. 28 for $t = 0.5$. Note a sample size of 2000 is necessary to bring the error down to below 2 percent, which is consistent with the effect of the pseudorandom number generator described earlier.

The error percentage due to the use of the difference algorithm (Heun in this case) only is also superimposed on the figure. This error is computed by comparing the sample statistics of the response samples obtained using the difference method with those of the exact response; i.e., $x(t) = \exp(w(t) - 0.5t)$, where $w(t)$ is a sample of the Wiener sequence generated. This error represents the contribution to the total error due to differencing, and is plotted as dashed lines. With reference to Fig. 28, it is obvious that the error due to differencing is small compared with that due to sample-to-sample variations. The latter is influenced greatly by the effectiveness of the pseudo-random number generator, and more work needs to be done to quantify this effect.

Higher-order equations--To show the generality of the difference approach to stochastic differential equations, several standard second-order equations are examined. These include the Langevin equation, which can be considered either as a second-order system on the displacement or a first-order system on the velocity. Convergence, accuracy and sample statistical issues are investigated. The results are similar to those just described, and details can be found in Ref. 22.

FINITE DIFFERENCE SOLUTIONS OF RANDOM DIFFERENTIAL EQUATIONS

The response of stochastic systems, i.e., systems which are governed by stochastic equations with Wiener or white noise processes, is Markovian. This property leads to many mathematical simplifications in analysis. For example, the transition probability density function satisfies the Fokker-Planck equation associated with the stochastic differential equation. The moments of the response are governed by certain deterministic equations, which are also well-known. Although few exact solutions to the Fokker-Planck equations have been found, and the
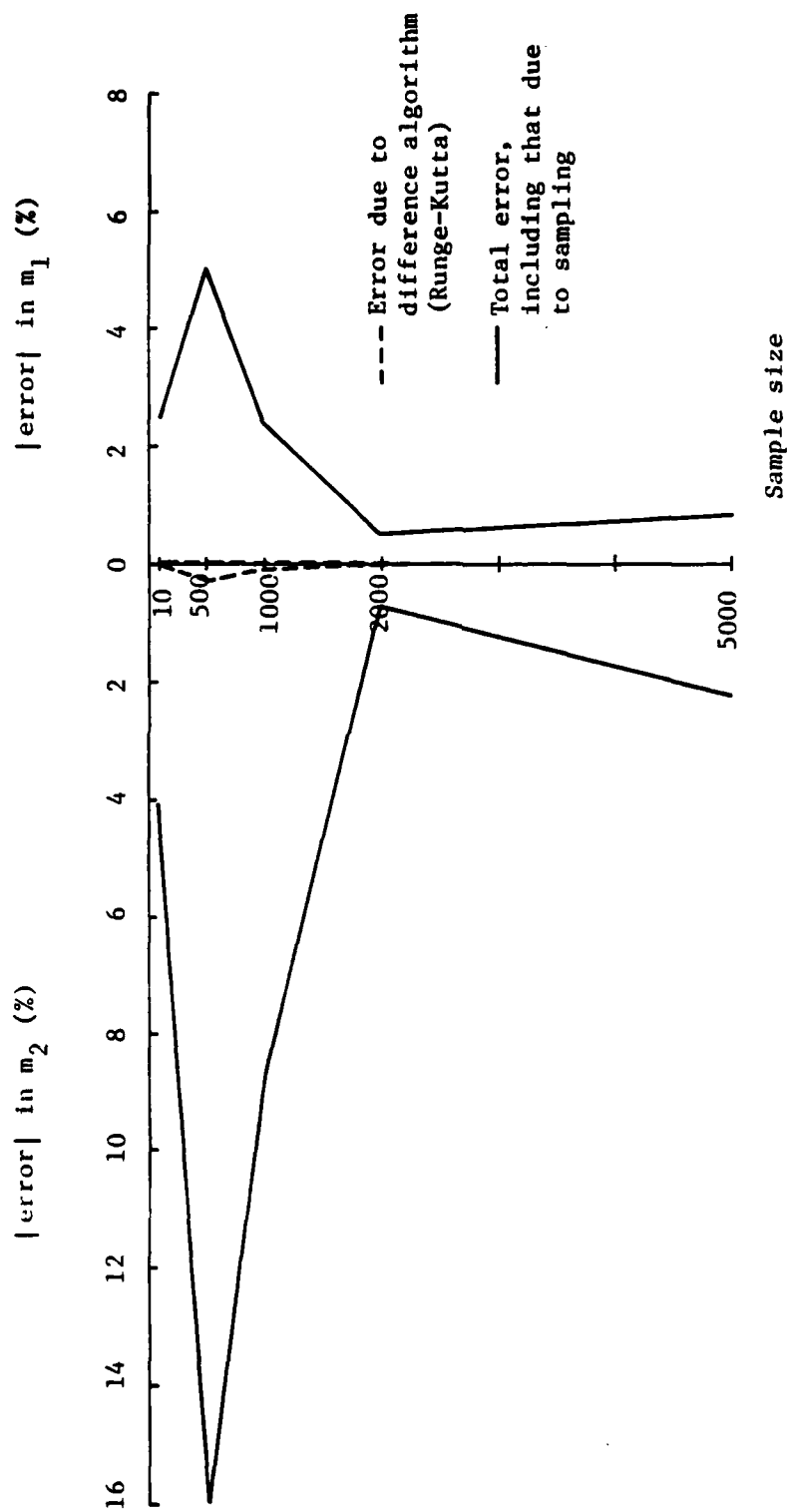
Figure 28. Comparison of exact moments and sample statistics, t = 0.5.

moment equations can become unvieldy, these analysis tools have
led to better understanding of the behavior of stochastic systems
(see Ref. 21). On the other hand, as we have shown in the
previous subsection, the applications of numerical techniques such
as finite difference to these stochastic differential equations
is complicated by the unique properties of the Wiener process and
the associated stochastic integral. The same properties which
permit simplification in closed-form analysis of the transition
probability and moments are causing difficulties in numerical
analysis.

A Wiener process or its formal derivative, the white noise
process, is an idealization. It represents one extreme of random
behavior, viz., completely erratic behavior. The sample at one
instant of time is not related to samples at any other times.
The other extreme of random behavior corresponds to complete
correlation, i.e., the process reduces to a random (constant)
variable. In the case of equations involving random constants,
some simplification in analysis is also possible, as described in
Ref. 21.

Physical processes generally have random coefficients which
fall somewhere between these two extremes. This general system
type is governed by differential equations with random process
coefficients, or simply, random differential equations. These
equations prove to be very difficult to solve. No general solu-
tion exists except for very simple equations.

The use of finite difference techniques to solve random
differential equations is investigated as part of the study. The
investigation is initialy limited to first-order systems. Our
goal is to study the behavior of the response in terms of numer-
ically computed first and second moments. It is noted that in
this application, the difficulty presented by the numerical
approximation of the stochastic (Ito and Stratonovich) integral
is avoided. However, in its place is the difficulty presented by
the need to generate numerically random processes which conform
to certain prescribed characteristics, such as mean and correla-
tion functions. This and other important aspects of the work are
summarized in the following. Details are given in Ref. 24.

Basic problem--To assess the feasibility of the finite dif-
ference technique in random differential equations, consider

$$x_t' + a_t x_t = b_t \quad , \quad x_t(0) = x_0 \qquad (42)$$

where a_t and b_t are random processes and x_0 may be a random variable or deterministic. Under certain conditions on a_t, the solution can be written formally by direct quadrature as (see Tikhonov, Ref. 31)

$$x_t = x_0 \exp\left[-\int_0^t a_s \, ds\right] + \int_0^t b_w \exp\left[-\int_0^t a_s ds\right]dw \qquad (43)$$

which is valid for deterministic x_0. Note the nonlinear dependence of x_t on a_t, even though Eq. 42 is linear. This nonlinear relationship is characteristic of problems of this type and is the cause of the difficulty encountered in the analysis.

When a_t and b_t are stationary, Gaussian processes with the following characteristics,

$$\langle a_t \rangle = m_1, \quad \langle [a_t - m_1][a_{t+\tau} - m_1]\rangle = \sigma_1^2 R_1(\tau)$$

$$\langle b_t \rangle = m_2, \quad \langle [b_t - m_2][b_{t+\tau} - m_2]\rangle = \sigma_2^2 R_2(\tau) \qquad (44)$$

$$\langle [a_t - m_1][b_{t+\tau} - m_2]\rangle = \sigma_1 \sigma_2 R_{12}(\tau)$$

the mean response process can be obtained from Eq. 43 (after much algebra, which is given in Tikhonov) as follows:

$$\langle x_t \rangle = x_0 \exp\left[-m_1 (t-0) + \frac{1}{2} \sigma_1^2 \int_0^t \int_0^t R_1(u_2 - u_1) \, du_1 du_2\right]$$

$$\qquad (45)$$

$$+ \int_0^t \left[m_2 - \sigma_1 \sigma_2 \int_0^t du_1 \int_0^t du_2 \int_v^t R_1(u_2 - u_1) R_{12}(w - u_3) \, du_3\right]$$

$$x \exp\left[-m_1(t - v) + \frac{1}{2} \sigma_1^2 \int_v^t \int_v^t R_1(u_2 - u_1) \, du_1 du_2\right] dv$$

The second moment $\langle x_t x_t \rangle$ can also be derived after even more algebra.

Equation 42 is also considered by Elrod (Ref. 32). He gave a solution under less stringent conditions on x_t and b_t, viz., x_t

can be random, and b_t is arbitrary, except its first two moments are given. Denoting the mean and correlation of a_t and b_t by $<a_t>$, $R_1(t, t')$ and $<b_t>$, $R_2(t,t')$, respectively, the first two moments are

$$< x_t > = \int_0^t \exp\left[\hat{Z}(0,t,0,u)\right] <b_u> du$$
$$+ <x_0> \exp\left[\hat{Z}(0,t,0,0)\right] \quad (46)$$

$$< x_t x_{t'} > = \int_0^{t'}\int_0^t \exp\left[\hat{Z}(t,t',u,v)\right] R_2(u,v) \, dudv$$

$$+ <x_0> \int_0^{t'} \exp\left[\hat{Z}(t,t',0,v)\right] <b_v> dv \quad (47)$$

$$+ \int_0^t \exp\left[\hat{Z}(t,t',u,0)\right] <b_u> du + <x_0^2> \exp\left[\hat{Z}(t,t',0,0)\right]$$

where

$$\hat{Z}(t,t',u,v) = -\int_v^{t'} <a_s> ds - \int_u^t <a_s> ds + \frac{1}{2}\int_v^{t'}\int_v^{t'} R_1(\sigma,\beta)\, d\sigma d\beta$$

$$+ \int_v^{t'}\int_u^t R_1(\sigma,\beta)\, d\sigma d\beta + \frac{1}{2}\int_u^t\int_u^t R_1(\sigma,\beta)\, d\sigma d\beta \quad (48)$$

The basic problem of Eq. 42 is selected for consideration mainly because the exact (moment) solutions are known, as shown. The integrals in Eqs. 46-48 can be evaluated analytically or numerically to any degree of accuracy desired. They will serve as a reference in evaluating the accuracy and convergence characteristics of the approximate solutions by the difference techniques. Note application of the difference techniques is not limited to Eq. 42, which is a linear equation and corresponds to highly idealized systems. Once the usefulness of the numerical techniques is estabilished using Eq. 42, they can be applied to very general random differential equations which do not have known analytical solutions.

Difference algorithms--As before, the finite difference approach involves replacing the differential operator d(.) in Eq. 42 with the difference operator $\Delta(.)$. Unlike the previous

application to stochastic differential equations, the implementation here is straightforward since there are no stochastic integrals to contend with. Consequently, all difference algorithms used in deterministic studies can be used here. The numerical studies to be described later are performed with a single-step, Runge-Kutta method, commonly known as the 1/8th rule.

Correlated random processes--To complete the difference equation formulation, it is necessary to generate on the computer nonstationary random processes which have certain given statistics (mean and covariance). Specifically, let $x(t,w)$ represent the process to be simulated. The mean and covariance are defined as

$$m(t) = <x(t,w)>$$

$$cov_x (t, t') = <[x(t,w) - m(t)] [x(t',w) - m(t')]$$

(49)

and these are prescribed. The objective is to generate numerically a random process with these prescribed first- and second-order statistics.

Computationally, the problem can be stated as follows. Let t_1 be the initial time Δt the time step, and n denote the number of times the process is to be observed. The time vector T can be written as

$$T_n = \begin{Bmatrix} t_1 \\ t_2 \\ \cdot \\ \cdot \\ t_i \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ t_n \end{Bmatrix} \quad , \quad t_i = t_1 + (i-1)\Delta t \qquad (50)$$

Generated on the computer, the random process will have the form of a matrix $[X_n]$ where

$$[X_n] = \begin{bmatrix} x_{11} & x_{12} & \cdot & \cdot & \cdot & \cdot & x_{1m} \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ x_{n1} & x_{n2} & \cdot & \cdot & \cdot & \cdot & x_{nm} \end{bmatrix} \qquad (51)$$

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Each column of the matrix is a realization or sample function of
x(t,w), and each row is a random variable from the process where
entries are samplings from the random variable. Since there are
readily available methods to generate a set of n-independent
random variables, the problem reduces to transforming the n-
independent random variables into n-random variables which have
the mean vector

$$
A_n = \left\{ \begin{array}{c} m(t_1) \\ m(t_2) \\ \cdot \\ \cdot \\ \cdot \\ m(t_n) \end{array} \right\} = \left\{ \begin{array}{c} m_1 \\ m_2 \\ \cdot \\ \cdot \\ \cdot \\ m_n \end{array} \right\} \tag{52}
$$

and the covariance matrix

$$
[M_n] = \left[\begin{array}{ccc} cov_x(t_1,t_1) & \cdots & cov_x(t_1,t_n) \\ cov_x(t_2,t_1) & \cdots & cov_x(t_2,t_n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ cov_x(t_n,t_1) & \cdots & cov_x(t_n,t_n) \end{array} \right] \tag{53}
$$

The procedure used in our formulation is due to Adomian and Elrod
(Ref. 33). The method is based on the use of the conditional
probability distribution function for a multidimensional process,
but details will not be given here. They can be found in Ref. 33
or Ref. 24.

    Numerical examples--To illustrate the convergence and accu-
racy of the finite difference approach, numerical solutions to
the basic problem are obtained for several special cases. These
are described below. It is of interest to compare the computed
statistical measures of the response process such as the mean and
covariance with the reference (exact) values.

    The overall implementation strategy follows that in Ref. 32.
Realization of the random processes a_t and b_t are generated
according to the procedure described in Ref. 33. These processes
have the prescribed mean and covariance properties which will be
different for the cases studied. When the initial condition x_0
is also random, realizations of x_0 are also generated. For the
i^{th} realization, one has

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$$\left\{\begin{array}{c} a_{1i} \\ a_{2i} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{ni} \end{array}\right\} \quad , \quad \left\{\begin{array}{c} b_{1i} \\ b_{2i} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_{ni} \end{array}\right\} \qquad (54)$$

where n is the number of time steps in the interval of interest. The difference algorithm is then used to obtain the corresponding realization of the response process

$$x_n = \left\{\begin{array}{c} x_{1i} \\ x_{2i} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{ni} \end{array}\right\} \qquad (55)$$

This process is repeated for m realizations to yield three (n x m) matrices

$$\left[\begin{array}{ccc} a_{11} \cdot & \cdots & \cdot a_{1m} \\ a_{21} \cdot & \cdots & \cdot a_{2m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} \cdot & \cdots & \cdot a_{nm} \end{array}\right] \, , \, \left[\begin{array}{ccc} b_{11} \cdot & \cdots & \cdot b_{1m} \\ b_{21} \cdot & \cdots & \cdot b_{2m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{n1} \cdot & \cdots & \cdot b_{nm} \end{array}\right] \, , \, \left[\begin{array}{ccc} x_{11} \cdot & \cdots & \cdot x_{1m} \\ x_{21} \cdot & \cdots & \cdot x_{2m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_{n1} \cdot & \cdots & \cdot x_{nm} \end{array}\right] \quad (56)$$

The statistical measures of the response process can be estimated from the matrix [X], which are then compared with the closed-form solutions. In addition, the statistical measures of the two coefficient processes, a_t and b_t can be estimated from the matrix [A] and [B], respectively. They are compared with the prescribed moments to check for fidelity in the computer-generated processes.

Depending on the complexity of the prescribed mean and covariance functions, closed-form expressions for the response moments, Eqs. 46 and 47 may not be easy to obtain. In that event, the integrals in Eqs. 46 and 47 (single and double integrals) are evaluated using numerical quadratures. However, for

the present purposes, these moments can still be considered exact.

For this first example, the mean and covariance of a_t and b_t are assumed to be

$$\langle a(t) \rangle = 0.5, \quad \langle b(t) \rangle = 0.5 + \sin(2\pi t)$$

$$\text{cov}_a(t,t') = \text{cov}_b(t,t') = \exp(-|t-t'|) \tag{57}$$

i.e., a_t and b_t are Ornstein-Uhlenbeck processes. The first two moments of x_0 are

$$\langle x_0 \rangle = 1. \quad \langle x_0^2 \rangle = 4/3 \tag{58}$$

In computer implementation, x_0 is generated as a random constant, uniformly distributed between 0. and 2.

The comparisons of mean and variance as functions of time are shown in Fig. 29. Two sets of finite difference results are shown, together with the exact (quadrature) solution. They correspond to sample sizes of 1,000 and 10,000, respectively, and are included to show the effect of sample size on convergence and accuracy of the approximate method. Time is non-dimensionalized, in terms of the correlation time constant which is unity. With reference to Fig. 29, the error in approximation increases with time. The error growth is faster for the variance, as expected. Whereas the error can be decreased by taking more samples, which increases computational cost, it is noted that eventually the error will become unacceptable. The error growth is accentuated because the basic behavior of the solution is also exponential.

The mean response is compared with the response of the mean equation, i.e., the same equation with deterministic coefficients, $\langle a(t) \rangle$ and $\langle b(t) \rangle$, replacing the random process coefficients $a(t)$ and $b(t)$, respectively. This is shown in Fig. 30. The mean response and the deterministic response are different, and the difference between them increases with time. This phenomenon is of course a characteristic of equations with random coefficients. The effect of different correlation functions on the response is also investigated. The results are similar to those shown in Figs. 29 and 30 and, hence, will not be presented. They can be found in Ref. 24.
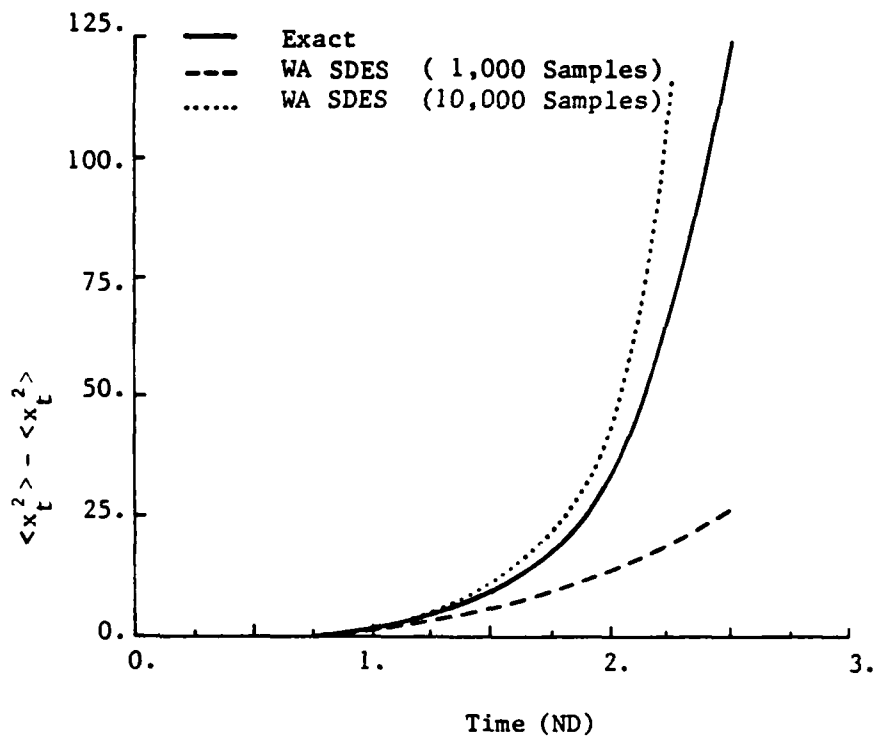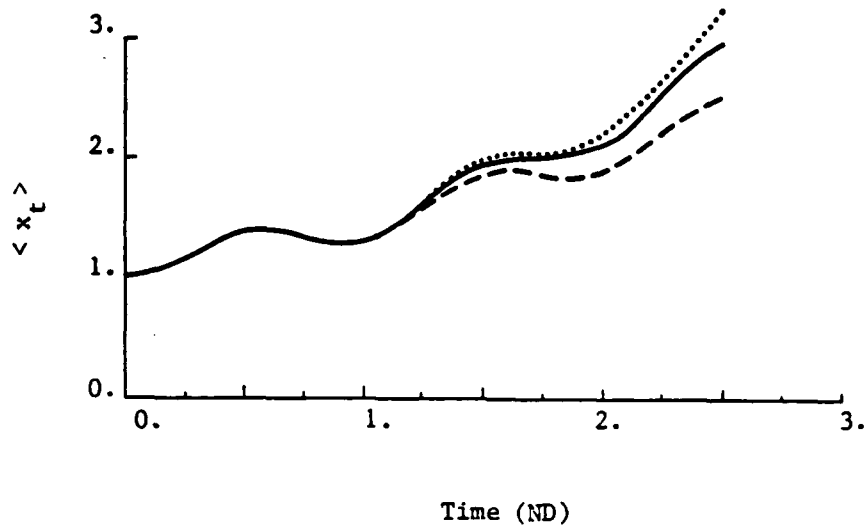
Figure 29. Mean and variance as functions of time, Case 1.
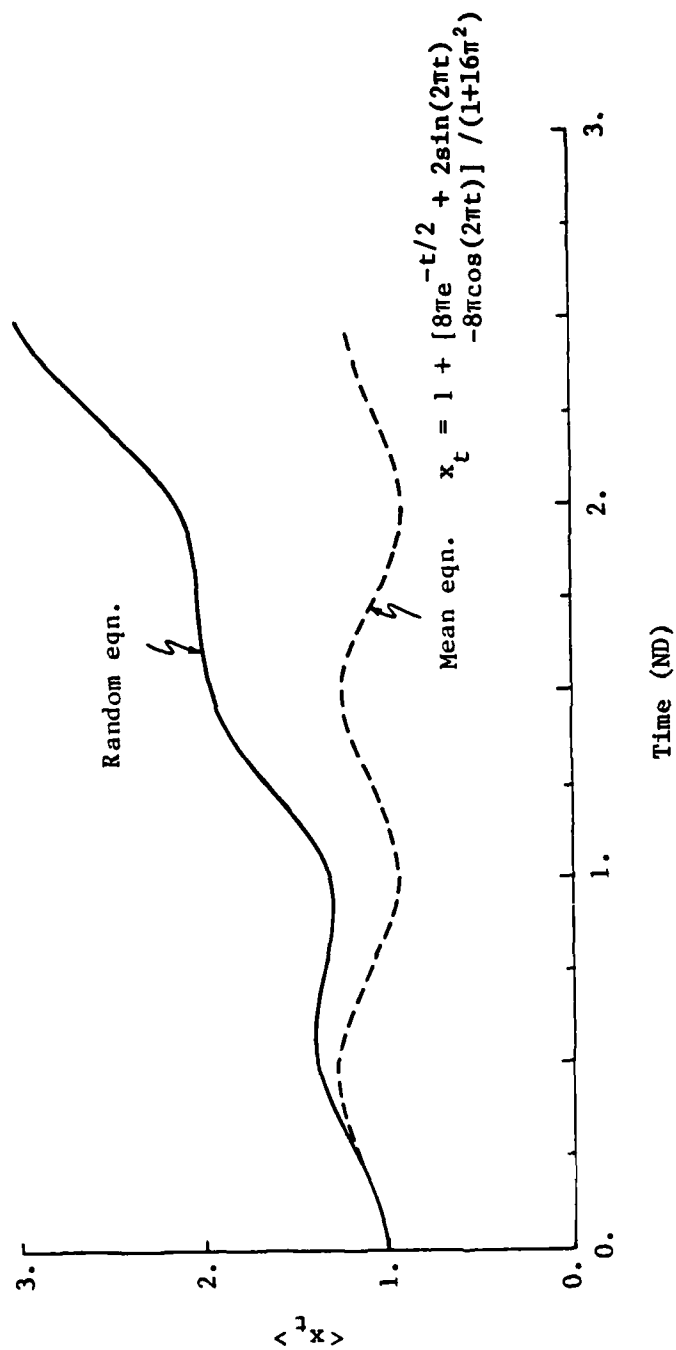
Figure 30. Comparison of mean of random solution and solution of "mean" equation.

$$x_t = 1 + [8\pi e^{-t/2} + 2\sin(2\pi t) - 8\pi\cos(2\pi t)] / (1+16\pi^2)$$

Random eqn.

Mean eqn.

Time (ND)

$\langle x_t \rangle$

The second example is used to illustrate the effect due to the random coefficient a_t only. The forcing function b_t is assumed deterministic and equal to unity, and the initial value x_0 is set to be 0. The statistical measures of a_t are

$$<a(t)> = 0.5, \quad \text{cov}_a(t,t') = k \exp(-|t-t'|) \quad (59)$$

The parameter k will be varied to change the magnitude of randomness in a_t.

The mean of the response process is compared in Fig. 31 with the exact mean. The curves denoted by k = 0 correspond to the deterministic response, i.e., solution to the equation when the mean value of the coefficients is used. With increasing k (randomness), the mean response increases more rapidly with time, and so does the error in the approximation. By comparing these curves with the curves for k = 0, an estimate of the influence of the random parameter on the mean response is obtained.

RANDOM VERSUS DETERMINISTIC

Consider Eq. 45 which gives the mean response of a first-order system with the random process coefficient subjected to random excitation. Randomness in the coefficient a_t, which changes the response characteristics of the system, is measured here by the correlation function $\sigma_1{}^2 R_1$. Randomness in the forcing function, b_t, is measured by the correlation function $\sigma_2{}^2 R_2$. The cross-correlation $\sigma_1 \sigma_2 R_{12}$ indicates the interplay between a_t and b_t, and will be assumed zero to expedite the following discussion.

It is interesting to consider several special cases of the solution given by Eq. 45. Suppose the randomness in the coefficient is ignored. Then, with $\sigma_1 = 0$, Eq. 45 gives,

$$<x_t> = x_0 \exp[-m_1(t-0)] + \int_0^t m_2 \exp[-m_1(t-v)] dv \quad (60)$$

Note that the mean solution does not depend on the randomness in the excitation. In other words, $\sigma_2{}^2 R_2$ does not appear in Eq. 60. It affects the dispersion in the response, however. Note also that Eq. 60 is identical to the deterministic mean solution; i.e., solution to the random equation when the random coefficient
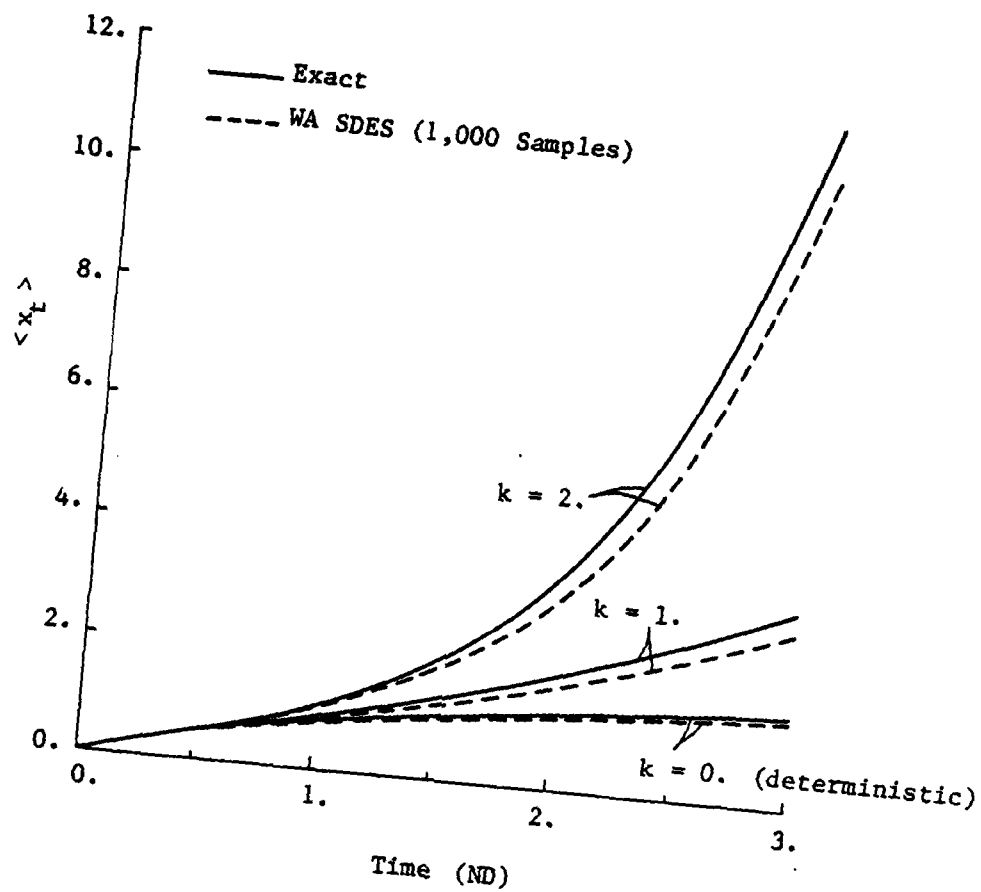
Figure 31. Comparison of exact and approximate means for several values of k.

is replaced by its mean value. In particular, it denotes the deterministic mean solution by $\bar{x}_t$. Then

$$\bar{x}_t' + m_1\bar{x}_t = m_2 \tag{61}$$

and

$$\bar{x}_t = x_0 \exp[-m_1(t-0)] + \int_0^t m_2 \exp[-m_1(t-v)]dv \tag{62}$$

Furthermore, the mean response is a linear function of the initial condition x_0 and the forcing function m_2, which is of course a well-known result in linear system and random vibration theory.

Suppose randomness in a_t is restored but the forcing function is deterministic, i.e. restore $\sigma_1^2 R_1$ and set $\sigma_2^2 R_2 = 0$. The mean response is then, from Eq. 45,

$$\langle x_t \rangle = x_0 \exp[-m_1(t-0) + \frac{1}{2}\sigma_1^2 \int_0^t \int_0^t R_1(u_2-u_1)du_1 du_2]$$

$$+ \int_0^t m_2 \exp[-m_1(t-v) + \frac{1}{2}\sigma_1^2 \int_v^t \int_v^t R_1(u_2-u_1)du_1 du_2]dv \tag{63}$$

Note Eq. 63 is not the same as the deterministic mean response given in Eq. 62. They are not equal as long as there is randomness in the coefficient, i.e., $\sigma_1 \neq 0$. This phenomenon is well-known for random operator problems, and was illustrated in a numerical example given previously.

Finally, let the coefficient be a random constant instead of a random process, i.e., $R_1 = 1$. Equation 45 becomes,

$$\langle x_t \rangle = x_0 \exp[-m_1(t-0) + \frac{1}{2}\sigma_1^2(t-0)^2]$$

$$t \int_0^t m_2 \exp[-m_1(t-v) + \frac{1}{2}\sigma_1^2(t-v)^2]dv \tag{64}$$

which is also different from the deterministic mean solution.

REMARKS

The use of finite difference techniques to solve stochastic and random differential equations has been described. Numerical results indicate that, while the approach is feasible, accuracy in the computation of the first and second moments requires a fairly large sample size. Convergence is relatively uniform, but details of the convergence behavior appears to depend heavily on the noise generator, i.e., the numerics of the pseudorandom number generator. Said another way, the error due to replacing the differential operator with the difference approximation algorithm is small compared with that due to sample-to-sample variation.

Hence, while the direct finite difference method is applicable to stochastic and random differential equations, computational expenses for computing moments may be quite high. For engineering applications, the response process itself (the sample path) is probably not as important or meaningful as the knowledge of the statistical measures.

Based on the results described in this section, there are two directions to follow in future research. One direction is to develop numerical algorithms which govern the propagation of moments. For stochastic equations, there are explicit moment equations based on the Markov property of the response process. Limitations of this approach are summarized in Refs. 20 and 21. For random equations, there are no known general moment equations and much more research is needed. Another direction is to seek improvement in current random number generators, and more efficient sampling techniques.

In principle, stochastic and random differential equations can be expressed as integral equations and, for stochastic equations, this is the more rigorous formulation. Discretization techniques have been applied to random integral equations (see Ref. 25), but they also rely on numerically generated random processes. Extension to highly nonlinear systems does not appear to be easy.

As far as S/V assessments are concerned, the study has established the feasibility of using numerical techniques to analyze problems involving stochastic and random loads (forcing functions) and initial conditions. The discussion of this sec-

tion focuses on the finite difference method and discrete systems, but similar results are obtained using finite element techniques on distributed systems, as described in Ref. 14. The advantage of discretization methods is that they apply to general nonlinear equations, as well as simple linear equations, albeit with more computational effort. A step-by-step approach considers the equations to be linear within each step.

The difficulties described in this section are associated with equations with random coefficients, or random operators. The difficulties are economic difficulties, the resolution of which awaits better and more efficient noise simulation techniques. Given the state-of-the-art, numerical techniques are viable solution techniques for random operator problems but they can be expensive solutions when multiple degree-of-freedom systems are considered.

Aside from feasibility and economics, a third issue should be addressed. This is the issue of physical interpretation of mathematical results. It is seen from Eqs. 60-64 that the mean response of a system with random coefficients is different from the system response with the mean coefficients. If the random coefficients correspond to random variations in, say, the structural properties, the above result implies that the average behavior of a number of nearly identical structures differs from the behavior of the average structure. An example of this paradox is given in Ref. 16, where it is shown that mathematically the most probable response of a population of undamped simple oscillators is a heavily damped simple oscillator. The question then arises as to how this damping can be interpreted in the real world of S/V assessment. A partial resolution of this question is sought in the fuzzy set representation of uncertainties, the subject of the next section.

VI. FUZZY UNCERTAINTIES AND EXPERT OPINIONS

Although fuzzy sets and fuzzy logic are used in this work to represent nonrandom uncertainties, including expert judgment and opinions, it should be remembered that there are other representations. A brief survey of these other modeling techniques is included in the following for this purpose. Needless to say, there is heated debate on the merits of the different modeling approaches and the merits of subjective probability versus fuzzy set theory in particular. It is not the intent here to add to this debate. It suffices to say that compared with fuzzy sets, all other theories are variants of Kolmogorov's probability theory, and are designed to answer the question of what is belief and how belief can be assessed.

The next subsection describes several elements of fuzzy set theory in the event that the subject may not be as familiar to the reader as probability theories and crisp sets. This introduction is not meant to be complete, and relies on examples rather than mathematics. Details can be found in the references cited.

The main portion of the section describes two major studies of fuzzy uncertainties with S/V applications: the modeling and analysis of uncertainties associated with analytical models, and the assessment of damage to structures. The emphasis of the discussion is on feasibility and methodology development. Actual case studies are in progress.

SURVEY OF OTHER THEORIES

The difference between uncertainty as a frequency of occurrence and as a result of induction is long recognized, almost from the beginning of modern probability theory. A central issue is belief and partial belief. Ramsey (Ref. 34) defined belief as the propensity to act, and developed a personal probability theory which came to be known as the Ramsey-DeFinetti-Savage theory (Ref. 35). This theory, in turn, leads to ratio-scaled probability which satisfies the Kolmogorov axioms. However, in actual case studies, the theory is found lacking when compared with human behavior (see Hogarth, Ref. 36). Hence, it is sometimes referred to as the theory of the rational (or perfect) man.

Recognizing that probability theory does not conform to the way people think and behave and that the rational man must be trained to be a fallible human, leads to the relaxing of some of the axioms of probability. This then becomes the theory of weak-ratio probability. Some examples are the works by Dempster and Shafer, and by Wolfensen and Fine (Refs. 37 and 38). They have also become known collectively as the theory of subjective probability, but are not widely used in practice. The only exception is Shafer's theory of evidence, which has recently found its way into a number of expert systems.

ELEMENTS OF FUZZY SETS

Basic concepts--In Section III, the concept of fuzzy sets is introduced as a generalization of the crisp set. The characteristic function which defines a crisp set based on the binary yes-or-no proposition is generalized. A fuzzy set is represented by a membership function which corresponds to the degree of belongingness of an element x in the set $\underset{\sim}{A}$, i.e.,

$$\mu_{\underset{\sim}{A}}(x) = \alpha, \qquad 0 \le \alpha \le 1 \tag{65}$$

Since the membership function can have any value in the unit interval [0,1], the degree of belongingness varies from 0 to 1, or from completely does-not-belong to completely belongs. Of course, partial belongingness is possible and thereby fuzziness is represented.

Mathematically, fuzzy sets are defined as follows, given a universe S, which is the ensemble of all possibilities being considered, a fuzzy (sub)set $\underset{\sim}{A}$ of S is expressed by the membership function $\mu_{\underset{\sim}{A}}(x)$, which maps a point x in S to a value in the interval [0,1]; i.e., it gives the degree of belongingness that the element x is considered to be in the set $\underset{\sim}{A}$. Hence, $\underset{\sim}{A}$ is written as

$$\underset{\sim}{A} = \sum_{i=1}^{N} \frac{\mu_{\underset{\sim}{A}}(x_i)}{x_i} \quad \text{or} \quad \int_{S} \frac{\mu_{\underset{\sim}{A}}(x)}{x} \tag{66}$$

Note that in this traditional notation, the summation or integral sign should be interpreted as the union, and the horizontal bars and slanted slashes are used to emphasize the correspondence between the element x and its membership (x) in the set. They

are not divisions, although such confusion often arises.

Some examples will make this notation clear. Suppose the universe S is the set of all positive integers 1, 2, 3, 4, ... The crisp set A = 3, or the statement, in a physical context, that the concrete strength is 3 MPa, is

$$\text{equals } 3 = \frac{0.}{1} + \frac{0.}{2} + \frac{1.}{3} + \frac{0.}{4} + \frac{0.}{5} + \ldots \qquad (67)$$

The element 3 is given a membership 1 while all other elements have membership 0. The fuzzy set $\underset{\sim}{A}$ = approximately 3, on the other hand, may have the representation

$$\text{approximately equals } 3 = \frac{0.2}{1} + \frac{0.5}{2} + \frac{1.}{3} + \frac{0.5}{4} + \frac{0.1}{5} + \ldots \qquad (68)$$

Elements such as 2 and 4 belong to the set "approximately 3" but with a membership less than that of element 3. Equation 68 can be used to represent the statement that the concrete strength is approximately 3 MPa, for example.

Other examples can be readily given. A crisp stress-strain relation is given in Fig. 32a, and a fuzzy stress-strain relation in Fig. 32b. A crisp fragility curve and a fuzzy fragility curve are given in Fig. 33. When S/V assessment is based on degree of damage rather than fragility, a fuzzy set representation of damage states can be used, such as that given earlier in Fig. 20. The corresponding representation in crisp sets is not as obvious.

It is clear that one of the most important, if not the most important, element in the fuzzy set representation is the membership function. It is the essence of the fuzzy model of uncertainty which has been alluded to in Section III. The determination of the membership function is currently an active research area, and there is not enough space here to go into details. It is mentioned simply that the membership can be determined based on any data base, objective or subjective, statistical or otherwise. For instance, expert opinions can be used to define the membership, as will be shown in the next section. The generalization of the crisp set in the form of the membership function allows a wider class of data bases to be accepted and represented without causing undue mathematical inconsistency.
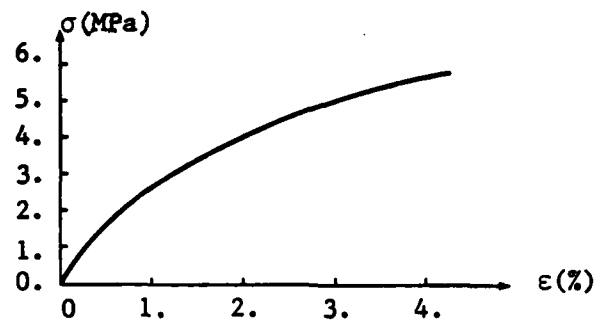
(a) Crisp stress-strain law
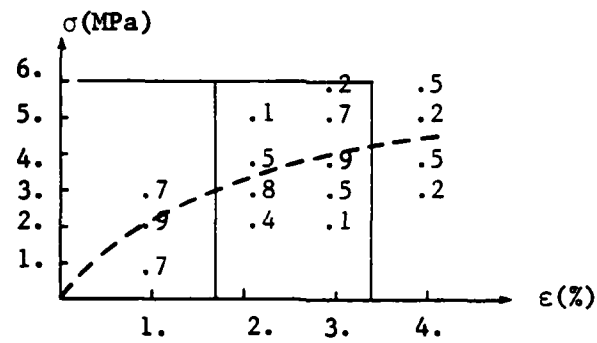
(b) Fuzzy stress-strain law

Figure 32.   Crisp and fuzzy constitutive relationships.
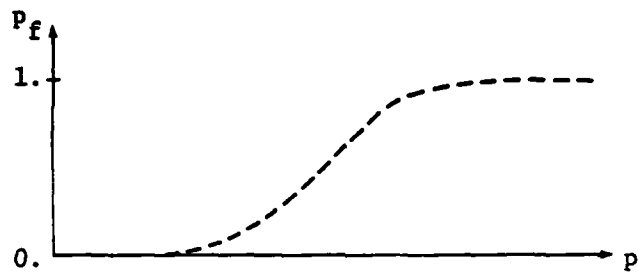
(a) Crisp fragility curve
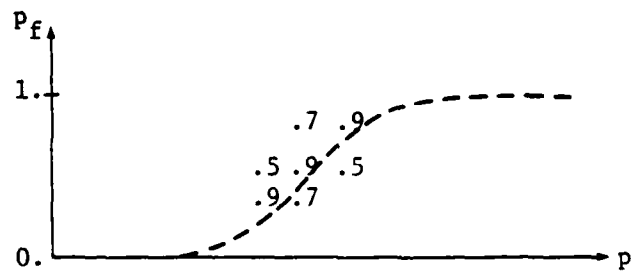
(b) Fuzzy fragility curve

Figure 33. Crisp and fuzzy fragility curves.

Mapping and transformation--Mapping and transformation of
fuzzy sets are not very different from mapping and transformation
of crisp sets. In fact, there is an established principle,
called the extension principle, by which the former can be per-
formed using operations of the latter. This is summarized below.
In this discussion, mapping and transformation of fuzzy sets can
be considered as the propagation of fuzzy uncertainties in
dynamic system response, in much the same way that random uncer-
tainties are propagated in random and stochastic equations. For
the time being, the system dynamics is considered crisp. The
response is fuzzy because the initial conditions, forcing func-
tions, or coefficients of the equations are fuzzy. The case when
the system dynamics is also fuzzy is taken up in the next subsec-
tion when fuzzy relations are discussed.

Let a fuzzy set $\underset{\sim}{A}$ be defined on the universe X. Further-
more, let a point x in X be mapped into a point y in another
universe Y by the transformation $y = f(x)$. The image of $\underset{\sim}{A}$ is
then a fuzzy set $\underset{\sim}{B}$ defined on Y, given by

$$\underset{\sim}{B} = \int_Y \frac{\mu_{\underset{\sim}{B}}(y)}{y} \tag{69}$$

where

$$\mu_{\underset{\sim}{B}}(y) = \mu_{\underset{\sim}{A}}(x) \Big|_{y = f(x)} \tag{70}$$

This mapping is sometimes abbreviated as $\underset{\sim}{B} = f(\underset{\sim}{A})$, although its
true meaning as defined in Eqs. 69 and 70 may be obscured or misin-
terpreted. It should be remembered that $\underset{\sim}{B}$ is not a function of $\underset{\sim}{A}$,
but rather that y is a function of x and the membership of x is
transported to be the membership of y. An illustration of the
mapping operation is given in Fig. 34.

When the transformation $y = f(x)$ is not a one-to-one mapping
but, say, a many-to-one, the membership of $\underset{\sim}{B}$ is obtained by
taking the maximum of all the memberships of the x's which are
mapped onto the particular value of y. Mathematically, this is
denoted by

$$\mu_{\underset{\sim}{B}}(y) = \sup_{\substack{x \in X \\ y=f(x)}} \left\{ \mu_{\underset{\sim}{A}}(x) \right\} = \bigvee_{\substack{x \in X \\ y=f(x)}} \left\{ \mu_{\underset{\sim}{A}}(x) \right\} \tag{71}$$

where sup denotes the supremum operation and is the same as the
maximum operation (denoted by $\vee$) in this case. The operation is

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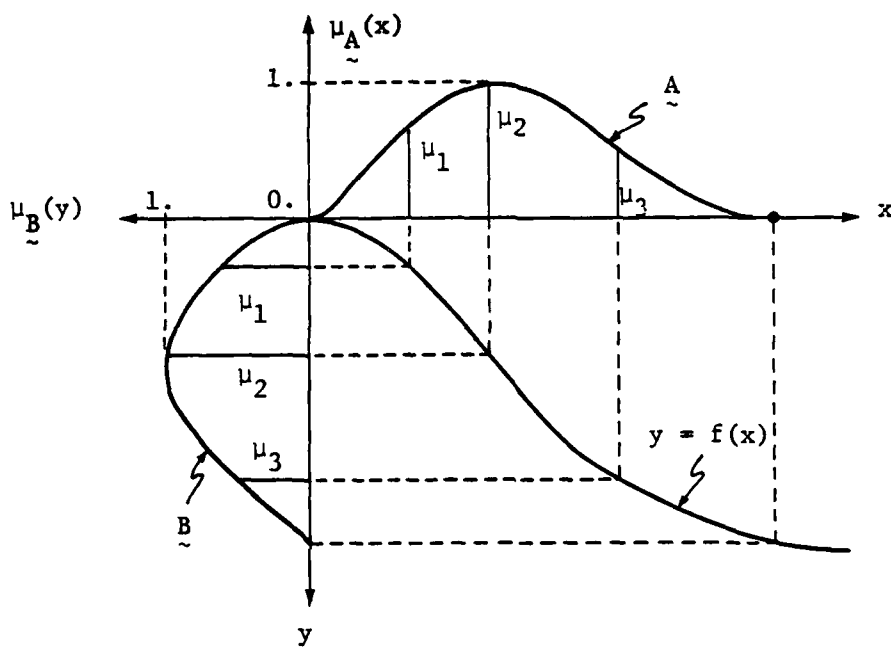
Figure 34.   Transformation of fuzzy sets A to B under
              mapping y = f(x)--mapping is one-to-one.
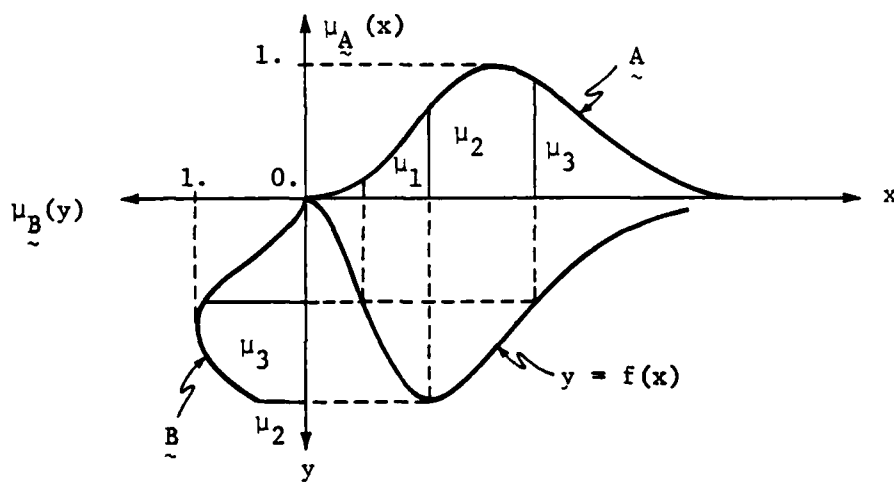
Figure 35.   Transformation of fuzzy sets A to B under mapping $y = f(x)$--mapping is many-to-one.

illustrated in Fig. 35. This idea can be extended to a function of many independent variables, and in fact to general functions by Zadeh's extension principle. Details are in Ref. 9.

Fuzzy relations--To introduce the concept of fuzzy relations, consider the fuzzy stress-strain relation of Fig. 32 mentioned earlier. Suppose a crisp value of the strain is selected, say, $\varepsilon_1 = 1$ percent. With reference to the figure, three possible values of σ correspond to $\varepsilon_1 = 1$, viz., 1., 2., and 3. MPa. Furthermore, σ takes on the value 1. MPa with membership 0.7, the value 2. ksi with membership 0.9, and the value 3. MPa with membership 0.7. Hence, the image of the crisp ε_1 is

$$\sigma_1 = \frac{0.7}{1.} + \frac{0.9}{2.} + \frac{0.7}{3.} \tag{72}$$

What is the image of ε_1 when ε_1 is itself fuzzy? This is the subject of fuzzy relations.

A relation relates at least two quantities, A and B. Call this relation R, as illustrated in Fig. 36. When R is crisp, e.g., mappings given in Fig. 34, a crisp A will map into a crisp B. A fuzzy $\underset{\sim}{A}$ will map into a fuzzy $\underset{\sim}{B}$, as described earlier. Suppose R is now fuzzy: a crisp A will still map into a fuzzy $\underset{\sim}{B}$, as illustrated by the stress-strain example. When $\underset{\sim}{A}$ is fuzzy and R is fuzzy, $\underset{\sim}{B}$ is also fuzzy. The computational operations are described below in the context of the fuzzy stress-stain relation.

Given a fuzzy $\underset{\sim}{\varepsilon}$ and the fuzzy stress-strain relation R, the corresponding fuzzy stress $\underset{\sim}{\sigma}$ is

$$\underset{\sim}{\sigma} = \underset{\sim}{\varepsilon} \circ \underset{\sim}{R} \tag{73}$$

where o is called the composition operation defined by

$$\mu_{\underset{\sim}{\sigma}}(\sigma) = \underset{\sigma}{V} [\mu_{\underset{\sim}{\varepsilon}}(\varepsilon) \wedge \mu_{\underset{\sim}{R}}(\sigma,\varepsilon)] \tag{74}$$

and where V is the maximum operation and $\wedge$ is the minimum operation, respectively. As an aid only in remembering the rules of the composition operation, one can use the analogy in matrix
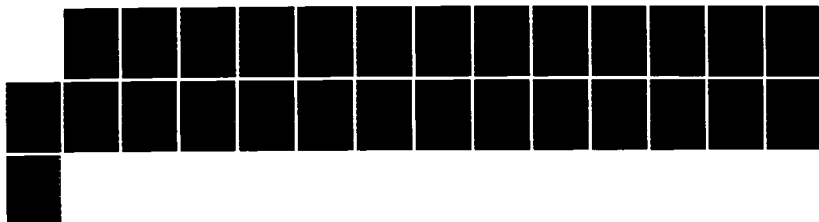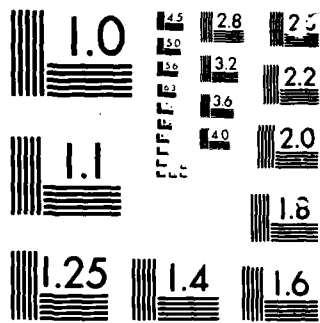
MICROCOPY CHART
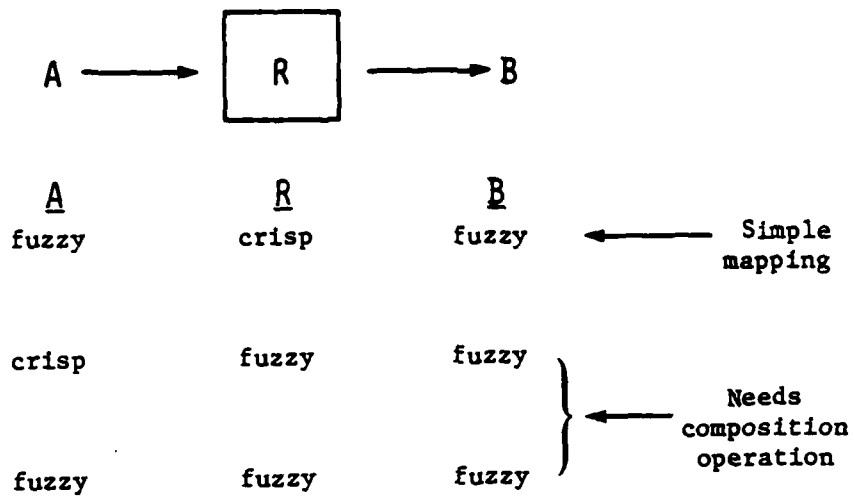
Figure 36. Fuzzy relations and propagation of fuzziness.

multiplication where

$$(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N) \ x \ \begin{bmatrix} r_{11} & \cdots & \cdots & r_{1M} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ r_{N1} & \cdots & \cdots & r_{NM} \end{bmatrix} \tag{75}$$

$$= [(\varepsilon_1 \ x \ r_{11} + \varepsilon_2 \ x \ r_{21} + \ldots), \ (\varepsilon_1 \ x \ r_{12} + \varepsilon_2 \ x \ r_{22} + \ldots), \ldots]$$

Multiplication is replaced by the minimum operation, and addition is replaced by the maximum operation. In the stress-strain example, if

$$\underset{\sim}{\varepsilon} = \frac{0.5}{2} + \frac{0.4}{3} \tag{76}$$

then the corresponding $\underset{\sim}{\sigma}$ is

$$\underset{\sim}{\sigma} = (0.5, \ 0.4) \ 0 \ \begin{bmatrix} 0.4 & 0.8 & 0.5 & 0.1 & 0. \\ 0.1 & 0.5 & 0.9 & 0.7 & 0.2 \end{bmatrix} \begin{matrix} (\varepsilon = 2) \\ (\varepsilon = 3) \end{matrix}$$

$$(\sigma = 2 \quad 3 \quad 4 \quad 5 \quad 6 \ )$$

$$\tag{77}$$

$$= \frac{[(0.5 \wedge 0.4) \vee (0.4 \wedge 0.1)]}{2} + \frac{[(0.5 \wedge 0.8) \vee (0.4 \wedge 0.5)]}{3} + \ldots$$

$$= \frac{0.4}{2} + \frac{0.5}{3} + \frac{0.5}{4} + \frac{0.4}{5} + \frac{0.2}{6}$$

A fuzzy relation is formed from a conditional relation such as: if $\underset{\sim}{A}_1$ then $\underset{\sim}{B}_1$, if $\underset{\sim}{A}_2$ then $\underset{\sim}{B}_2$, and so on where $\underset{\sim}{A}_i$ and $\underset{\sim}{B}_i$ are fuzzy statements. The mathematics involved is quite simple. Denote the relation corresponding to $\underset{\sim}{A}_i$ and $\underset{\sim}{B}_i$ by $\underset{\sim}{R}_i$. Then,

$$\underset{\sim}{R}_i = \underset{\sim}{A}_i \ x \ \underset{\sim}{B}_i \tag{78}$$

where the cross-product operation (x) is

$$\mu_{\underset{\sim}{R}}(x,y) = \int_{A_i \times B_i} \int \mu_{\underset{\sim}{A}_i}(x) \wedge \mu_{\underset{\sim}{B}_i}(y) /(x,y) \tag{79}$$

That is, $\underset{\sim}{R}_i$ is a two-dimensional array and the membership of its member $r_{\ell m}(x,y)$ is given by the minimum of the memberships for

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$A_i(x_\ell)$ and $B_i(y_m)$. The global relation R is obtained by the union of R_1, R_2, ... or $R = \sum_i R_i = \bigcup_i A_i \times B_i$.

Fuzzy relations are useful tools for modeling uncertain phenomena, mechanistic or otherwise. In the following subsections, it will be used in two applications as illustration. There are many other such tools provided by fuzzy set theory, but it is not possible to include them in this discussion.

STUDY IN MODEL UNCERTAINTIES

Some model uncertainties are nonrandom--In Section II, the uncertainties associated with an analytical model (in soil-structure interaction) was used as an example of nonrandom uncertainties. It was mentioned that a pitfall of the all-probabilistic approach to S/V assessment was to force such uncertainties to be random, which led to undesirable consequences.

Perhaps it should also be noted here that there are random model uncertainties. Empirical formulae based on regression analysis are classic examples. Uncertainties associated with the choice of the regression parameters are modeled as random variables in order to represent the scatter in the data. This is common practice in statistical analysis but, unfortunately, is also the source of much confusion; the statistical practice has been extended somewhat indiscriminately to all matters concerning models. Tell-tale signs of such possible misuse have been mentioned on several previous occasions and will not be repeated here. However, it may be helpful to look at the subject from a slightly different point of view.

Returning to the example on soil-structure interaction models, let us focus on how uncertainties in these models can possibly be assessed. Random experiments do not make sense here because only a few (three to four) models are considered. They do not differ from one another because of inherent heterogeneity. Neither is the selection of one model over another a matter of chance. Furthermore, the result of taking the average of all the predictions from the models does not correspond to any physical model. Certainly, the average obtained this way does not have the same meaning as a statistical average or expectation.

How models are evaluated--At present, model uncertainties
are evaluated by calibration. Controlled experiments are per-
formed to serve as the real-world reference. Models of this
phenomenon are postulated and used to simulate the experiments.
By comparing the model behavior with test data, the uncertainties
in the models are assessed, somehow. Details of this procedure
as applied to soil-structure interaction models is given in Ref.
4. A schematic diagram showing the procedure to evaluate material
models (soil, concrete) is given in Fig. 14.

Note that although the procedure appears straightforward,
it is far from complete. Many more questions are raised but not
answered. For example, what are controlled experiments and how
are these experiments determined? How do they relate to the real-
world phenomenon to be studied? How can uncertainties in the
source model, the mechanism model and the material model be
separated? Above all, what does a comparison between test data
and model response really mean, and how can the results from such
a comparison be used to quantify the uncertainties in the model?

Role of engineering judgment--To delve deeper into the sub-
ject of model uncertainty is to go beyond the scope of this
report. Suffice it to say that the questions posed are well-
acknowledged by the S/V community and the practice of calibration
is generally accepted despite these questions. The reason lies
in judgment; engineers find refuge from the unknowns by relying
on their judgment, which is based on related experience, general
knowledge, and subjectivity. It is used in all our evaluations,
and especially to compensate for sparse data and when extrapola-
tion beyond the data range becomes necessary. In terms of
models, experts are aware of the usage and shortcomings of cer-
tain types of models, from having worked with them in previous
applications or by previous comparison with data and other refer-
ences. They know that a particular feature of a model is essen-
tial to represent a certain phenomenon and, in the same manner,
they also know that another feature of the same model, when left
unchecked, will lead to erroneous predictions.

Fuzzy set approach--This study in fuzzy set attempts to
establish a framework by which subjective estimates of model
behavior based upon sparse data and considerable engineering
judgment can be incorporated into S/V assessment. The approach

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consists of four major components:

(1) Identify major features of the models and their
     gravities (G_i) and importances (I_i),
     i = 1, 2, ... N, where N is the total number
     of features. The curly underlines used previ-
     ously to denote fuzzy sets will be omitted from
     here on to simplify the notation.

(2) Combine the gravities and importances into a
     global fuzzy relation $P = \sum_i P_i = \bigcup_i G_i \times I_i$.

(3) Identify the relation between importance of feature I_i
     and its effect on the predicted response C_i.
     Summarize the relation as $R = \sum_i R_i = \bigcup_i I_i \times C_i$.

(4) Form the relation between the global character of the
     model features and their impact on the predicted
     response by F = P O R, i.e., the composition of the
     two relations P, R.

The procedure is described in detail in Ref. 5, together with
numerical examples and illustrations.

A typical result is the relation shown in Table 2, relating
gravity of a model feature with the correction factor to be
applied to the model. In this table, y is the theoretical res-
ponse of the system (e.g., deflection of slab) computed according
to the model. To show how this result can be used in practice,
suppose a particular feature of a particular model is judged to
have medium gravity where medium is defined as

$$\frac{0.2}{0.3} + \frac{0.6}{0.4} + \frac{1.}{0.5} + \frac{0.6}{0.6} + \frac{0.2}{0.7} \tag{80}$$

The correction to be applied to the model response is then

$$K(y) = (G = medium) \, O \, F$$

$$= \frac{0.5}{y} + \frac{0.5}{1.1y} + \frac{1.}{1.2y} + \frac{0.33}{1.3y} + \frac{0.2}{1.4y} \tag{81}$$

TABLE 2. FUZZY RELATION BETWEEN GLOBAL GRAVITY OF MODEL
FEATURE AND ITS IMPACT ON THE MODEL RESPONSE Y

G	C (Model Correction Factor)				
(Gravity)	y	1.1y	1.2y	1.3y	1.4y
0.3	0.2	0.2	0.2	0.2	0.2
0.4	0.5	0.5	0.6	0.33	0.2
0.5	0.5	0.5	1.0	0.33	0.2
0.6	0.5	0.5	0.6	0.33	0.2
0.7	0.2	0.2	0.2	0.2	0.2
0.8	0.2	0.2	0.5	0.2	0.2
0.9	0.2	0.2	0.9	0.2	0.2
1.0	0.2	0.2	1.0	0.2	0.2

That is, the response is approximately 20 percent higher than
that predicted by the model when the effects of modeling uncer-
tainty are taken into consideration.

STUDY IN DAMAGE ASSESSMENT

Damage states are not clear-cut--Another important source of
nonrandom uncertainties mentioned in Section III concerns the
assessment of damage. In many fields of engineering, damage and
its interpretation are not clear-cut. This is especially true
for protective structures because they are heavily reinforced and
yet are expected to be loaded into severe damage and even total
collapse. Only a limited number of tests can be performed, and
the tests are usually done on small-scale structures using simu-
lated loadings. A sample damaged specimen is shown in Fig. 37.
The evaluation of light, medium and severe damages differs from
one expert to another. The damage ranges are expected to over-
lap, i.e., damage does not change abruptly from light to medium,
and from medium to severe upon reaching certain crisp thresholds.
Other factors such as scarcity of data and the need to extrapo-
late the data to realistic loading, full-size prototypes and
imperfect structures add much more complexity to the assessment
of damage.

Scope of damage assessment--The foregoing describes only a
small portion of the overall scope of damage assessment. In
addition to the damaged specimens, many other measurements
(active as well as passive) are available and need to be included
in the evaluation. An overview of the scope of damage assessment
is shown in Fig. 38, although the present study focuses on the
soft data portion of the assessment. Soft data refer to photo-
graphs and visual images of tested specimens, and they carry much
information which has not been explicitly quantified and identi-
fied before. The following discussion describes the approach
used to obtain experts' evaluations of the soft data, and to
aggregate these inputs into a form which can be readily incor-
porated into S/V assessment.
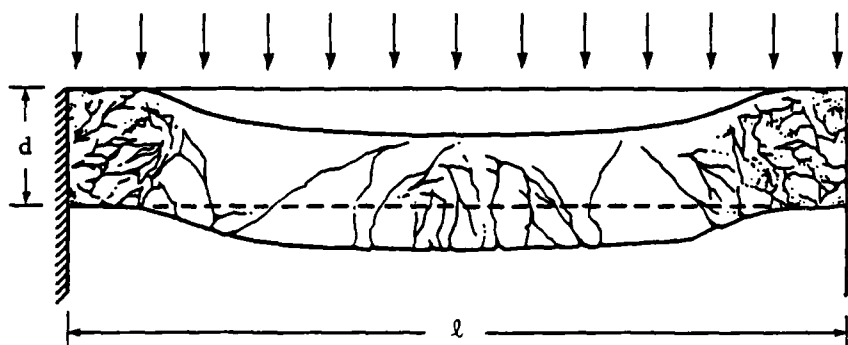
Figure 37. Example of damage in specimen observed after
test.
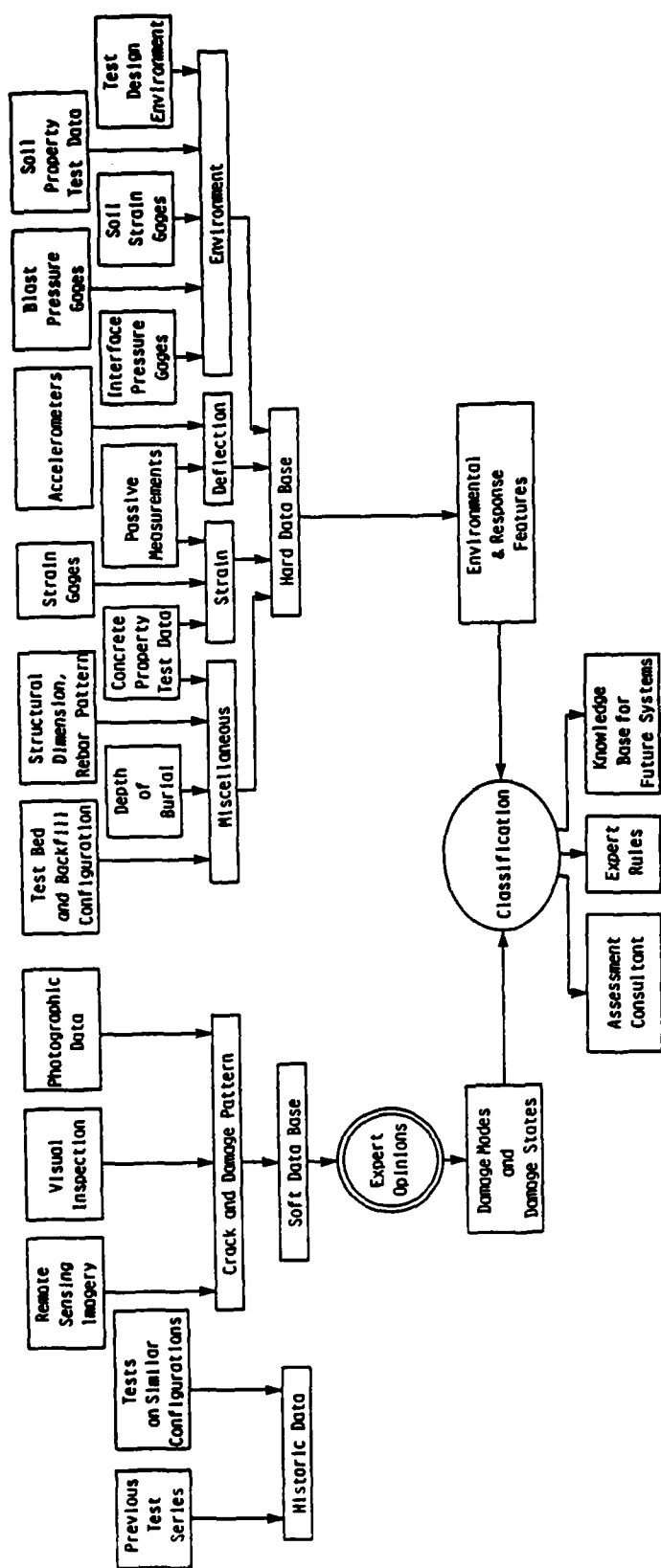
Figure 38. Approach used in case study of damage assessment.

Solicitation and aggregation of expert opinions--The problem
of solicitation of expert opinions can be described as follows.
An expert or several experts are shown a piece of data which can
be in the form of pictures, graphs, time-traces, etc. The
experts are then asked to give their assessments of the data. If
the data are denoted by A and the assessment from expert No. 1 is
B_1, the assessment from expert No. 2 is B_2, and so on, the result
can be expressed concisely as

$$\text{If A, then } B_1, B_2, \ldots \tag{82}$$

When there are more than one piece of data to be evaluated, the
result of solicitation is

$$\text{If } A_1, A_2, \ldots, \text{ then } B_1, B_2, \ldots \tag{83}$$

Here the same pieces of data $(A_1, A_2, \ldots)$ are shown to different
experts who then provide the assessments B_1, B_2, ..., respec-
tively.

Similarly, the process can involve the following. Expert
No. 1 is shown data A_1, and he gives the assessment B_1. Expert
No. 2 when shown data A_2 gives the assessment B_2, and so on. The
result can be expressed as

$$\text{If } A_1, \text{ then } B_1, \text{ and}$$
$$\text{If } A_2, \text{ then } B_2, \text{ and} \tag{84}$$
$$\ldots \ldots$$

Having obtained the expert opinions, the next step is to
combine them. This is the problem of aggregation. The assess-
ments B_1, B_2, ... are analyzed and synthesized in order to arrive
at some global or overall assessment. One usual outcome of
aggregation is a consensus assessment B, but it should be empha-
sized that consensus should not be the only goal of aggregation.
In fact, contradictory and ambiguous assessments should be right-
fully reflected in the aggregated product.

Solicitation and aggregation of expert opinions are widely
practiced in S/V assessment although they are not noted expli-
citly as such. For example, in every technical meeting, planning
session, workshop and test-related conference, experts are called

together so that their opinions can be solicited. Examples of
data A's are measured strains and deflection of structural ele-
ments, observed crack patterns, etc. Examples of assessment B's
are damage states, residual strength of structures, fidelity of
test, survivability, etc. It is common practice to let the
experts process their assessments mostly by voice votes. Without
exception, the end-result is a decision or consensus.

The solicitation and aggregation of expert opinions have
many important facets and this report will not be able to
include all of them. There are different ways to elicit opin-
ions, to weight the different ability and experience of the
experts, and to include one's personal bias. Ways to refine
opinions by feedback and iterative evaluation also have been
studied. The precision and form of the assessment is another
important factor, as is the largely unresolved question of how to
treat commonality of knowledge and data base which may have been
shared by the experts. The study performed here emphasizes two
facets of the problem: consensus and subjectivity. Specifically,
it is felt that most existing methods center around the need to
have a consensus opinion, and that this may not be correct.
There may be very good reasons why opinions vary, as is often
the case in S/V assessment. The process of aggregation should
include these diverse opinions and not alter them. Consensus
methods also rely on large sample populations, which is not the
case for most S/V applications. Expert opinions are seen as non-
random, especially when weights and bias factors are included.

Fuzzy set approach—The fuzzy set approach consists of the
following four major steps (see also Fig. 39) :

(1)   Groups experts into homogeneous subgroups;
(2)   Solicits opinions from members of each subgroup;
(3)   Aggregates the opinions of members of a subgroup,
      including weights on the opinions; and
(4)   Aggregates opinions of the subgroups.

The approach makes repeated use of two fuzzy techniques
called fuzzy classification and fuzzy identification. Fuzzy
classification is used to separate the experts into subgroups,
and the subgroups into sub-subgroups, if necessary. It is also
used to help define the weighting factors assigned to the sub-
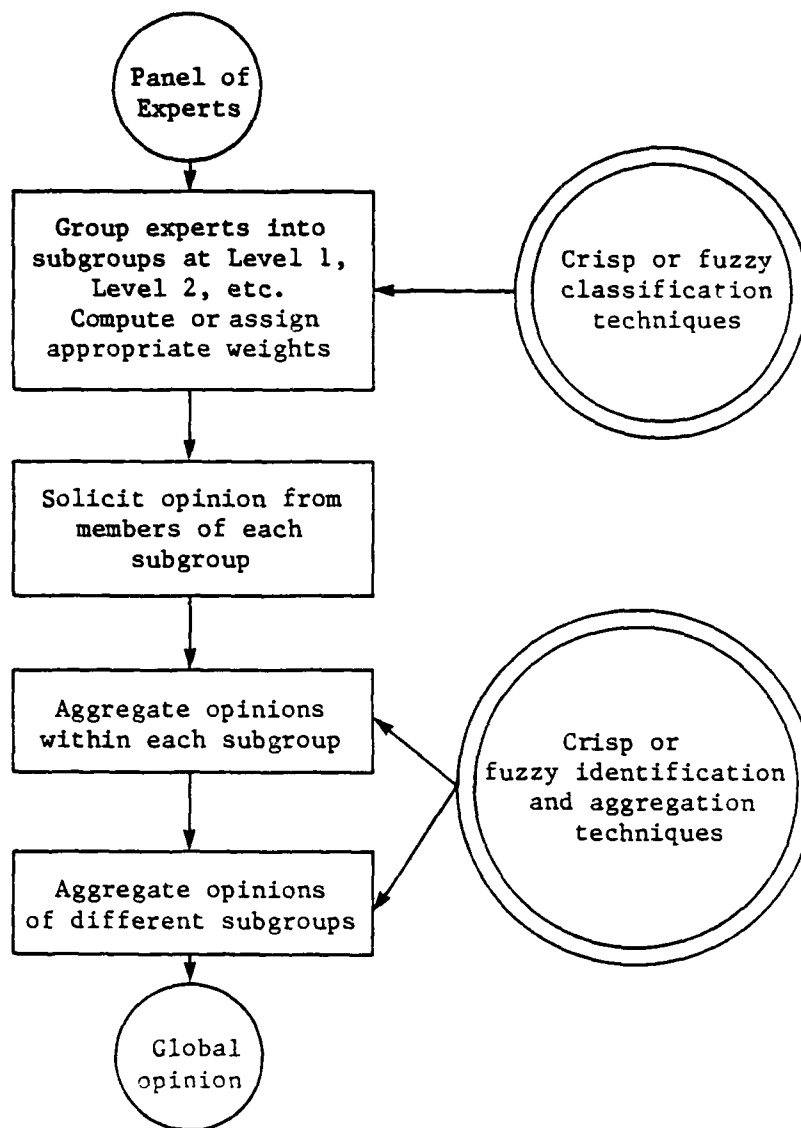
Figure 39. Major steps in classification and aggregation of expert opinions.

groups. Fuzzy identification is used to synthesize the different
inferences of members of a subgroup. Synthesis refers here to
aggregating different assessments as mentioned previously, or to
constructing a global inference machine which in some sense best
summarizes the thinkings of the experts.

The following discussion is limited to the aggregation part
of the study. A description of the classification techniques used
and their applications is given in detail in Ref. 39. It is
noted that fuzzy classification is also useful for treating hard
data encountered in S/V assessment, such as those shown in Fig. 38,
and for comparing sparsely populated waveforms and ill-defined
measurements.

Inference and identification--Consider the proposition R: if
A then B. Here, A is called the antecedent or cause, and B is
called the consequence or effect. The process of arriving at an
answer B given data A is an inference process. When referring to
the inferred opinions, the B's, aggregation means combining these
expert opinions into a joint opinion or inference. Note only the
consequent B's are directly involved in this aggregation ap-
proach. When referring to the inference processes themselves,
the R's, aggregation means combining these relations into a
global, representative relation. Both the antecedent A's and the
consequent B's are used in this latter approach, which shall be
referred to as identification. The name comes from identifying
the inherent relation which provides the best representation of
all the constituent relations.

The first meaning of aggregation is more conventional, and
methods for combining expert opinions have been extensively
studied. They will be summarized in the following with more
details given in Ref. 39. Emphasis of the present discussion is
on identification, since it is more general and appropriate for
S/V assessment applications.

Combining expert opinions--The method of combining expert
opinions depends on whether the information is cardinal (numeri-
cal) or ordinal (linguistic). When opinions are given in the
cardinal scale, there are many choices for combining them. For
more details, see Zimmermann (Ref. 40), Zysno (Ref. 41), or Wong
(Ref. 39). Some examples are max, min, algebraic mean and

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geometric mean. For example, if u_i, $i=1,2,...,m$ are different
fuzzy opinions of m experts and their relative weights are d_i,
with

$$\sum_{i=1}^{M} d_i = m \tag{85}$$

then the algebraic mean is

$$u_\theta = \frac{1}{m} \sum_{i=1}^{m} d_i u_i \tag{86}$$

where the summation is understood to follow Zadeh's extension
principle (Eq. 70). The geometric mean is

$$u_\theta = \left(\prod_{i=1}^{m} u_i^{d_i} \right)^{\frac{1}{m}} \tag{87}$$

For a homogeneous subgroup, the opinions of its members should be
very close to one another, and all these operators will yield
comparable results. When the opinions are diverse, as may occur
with different subgroups, the above operators may give very
different results.

Methods to combine opinions and weights when they are given
on a linguistic scale have been proposed by Buckley (Ref. 42) and
Schmucker (Ref. 43). The methods require some lengthy explana-
tion and are summarized in Ref. 39. Sample applications are also
given in said reference.

Identification--Suppose the data-opinion pairs are denoted
by A_i and B_i, $i=1,2,...,N$, and each pair corresponds to a rela-
tion R_i given by

$$R_i = A_i \times B_i \tag{88}$$

according to Eq. 77. Given R_i and A_i, to recover B_i requires the
composition operation of Eq. 73. The problem of identification
is to find a relation R which can represent the fuzzy data pairs
(A_i, B_i), $i=1,2,...,N$ in some optimal sense. In other words,
R is to represent a combination of the constituent relations R_i,
$i=1,2,...,N$, in some optimal sense.

One common approach to do so is due to Mamdani (Ref. 44), which is the union method mentioned earlier. In particular, the answer is given by

$$R = \underset{i}{U}R_i = \int\int_{X \times Y} \underset{i}{V} \mu_{R_i}(x,y)/(x,y) \qquad (89)$$

Since the union R of several R_i contains all the R_i, one can say that R is the aggregated relation. However, note that in general,

$$A_i \circ R = B_i' \; , \; B_i' \neq B_i \qquad (90)$$

Hence, union aggregation will __not__ return all the original data-opinion pairs (A_i, B_i).

The approach used here is to try to minimize the difference between B'_i and B_i, i.e., the aggregated relation should return the original data as best as possible, and in that sense is the best summary of the constituent relations. The objective function to be minimized depends on the measure of distance, i.e.,

$$Q = \sum_{i=1}^{N} d_i^2, \quad d_i = d_i(B_i', B_i) \qquad (91)$$

where d_i is a dissimilarity measure between the fuzzy sets B_i and B'_i. Different variants of the identification algorithm correspond to using different dissimilarity measures and schemes to minimize the objective function.

One method investigated uses the Euclidean distance for d_i and a minimization procedure based on a modified Newton iteration scheme. Details are documented in Refs. 39, 45 and 46. The result is a relatively simple recursive equation for the aggregated relation

$$R^{(n+1)} = R^{(n)} - \alpha_n \frac{\partial Q}{\partial r_{ij}^{(n)}} \qquad (92)$$

where $R^{(n)}$ and $R^{(n+1)}$ are the (n)th and (n+1)th iterations of the desired R, respectively. The gradient $\partial Q/\partial r^{(n)}{}_{ij}$ is given in terms of the data-opinion pairs (A_i, B_i), and the current value of r_{ij}. The length scale, α_n, depends on the number of iteration

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n, and a constant $\beta \geq 0$ chosen empirically to increase the rate of convergence and minimize local oscillations.

Illustrative example--This example is based on the previous discussion on modeling uncertainties (see also Ref. 5). The effect of modeling on the analytical estimate of the deflection of the roof-slab of a buried box is considered, resulting in the following conditional relation,

R^1: If E is large, then C is large, or else
R^2: If E is medium, then C is medium, or else
R^3: If E is small, then C is small

where E refers to the effect of modeling and C refers to the correction factor which must be applied to the analytical estimate y to account for modeling uncertainties.

The linguistic value of large, medium and small for the modeling effect are:

$$\frac{0.1}{0.6} + \frac{0.2}{0.7} + \frac{0.5}{0.8} + \frac{0.9}{0.9} + \frac{1.}{1.}$$

$$\frac{0.2}{0.3} + \frac{0.6}{0.4} + \frac{1.}{0.5} + \frac{0.6}{0.6} + \frac{0.2}{0.7} \qquad (94)$$

$$\frac{1.}{0.} + \frac{0.9}{0.1} + \frac{0.5}{0.2} + \frac{0.2}{0.3} + \frac{0.1}{0.4}$$

and those for C are defined as:

$$\frac{0.1}{y} + \frac{0.33}{1.1y} + \frac{0.55}{1.2y} + \frac{0.78}{1.3y} + \frac{1.}{1.4y}$$

$$\frac{0.1}{y} + \frac{0.16}{1.1y} + \frac{1.}{1.2y} + \frac{0.16}{1.3y} + \frac{0.1}{1.4y} \qquad (95)$$

$$\frac{1.}{y} + \frac{0.78}{1.1y} + \frac{0.55}{1.2y} + \frac{0.33}{1.3y} + \frac{0.1}{1.4y}$$

TABLE 3. COMBINED RELATION BETWEEN MODELING EFFECT AND ITS
         CONSEQUENCE ON MODEL PREDICTION, USING MAMDANI'S
         METHOD OF UNION

E \ C	y	1.1y	1.2y	1.3y	1.4y
0.0	1.00	0.78	0.55	0.33	0.10
0.1	0.90	0.78	0.55	0.33	0.10
0.2	0.50	0.50	0.50	0.33	0.10
0.3	0.20	0.20	0.20	0.20	0.10
0.4	0.10	0.16	0.60	0.16	0.10
0.5	0.10	0.16	1.00	0.16	0.10
0.6	0.10	0.16	0.60	0.16	0.10
0.7	0.10	0.20	0.20	0.20	0.20
0.8	0.10	0.33	0.50	0.50	0.50
0.9	0.10	0.33	0.55	0.78	0.90
1.0	0.10	0.33	0.55	0.78	1.00

TABLE 4. FINAL RESULT FROM IDENTIFICATION ALGORITHM

E \ C	y	1.1y	1.2y	1.3y	1.4y
0.0	1.00	0.78	0.55	0.33	0.10
0.1	0.90	0.78	0.55	0.33	0.10
0.2	0.50	0.50	0.50	0.33	0.10
0.3	0.10	0.16	0.20	0.16	0.10
0.4	0.10	0.16	0.60	0.16	0.10
0.5	0.10	0.16	1.00	0.16	0.10
0.6	0.10	0.16	0.60	0.16	0.10
0.7	0.10	0.16	0.20	0.16	0.10
0.8	0.10	0.33	0.50	0.50	0.50
0.9	0.10	0.33	0.55	0.78	0.90
1.0	0.10	0.33	0.55	0.78	1.00

The resultant relation using the union operation of aggregation is given as Table 3. In this example, the three if-then relations can be regarded as three alternatives from one expert, or three opinions from three experts. The identification procedure is transparent to the interpretation of the information.

When this matrix is used as the initial guess for R to start the identification algorithm with equal weights on the three experts, the result returned upon convergence is very close to its initial guess (see Table 4). This means that the initial guess is a good choice, and only a few iterations are necessary to reduce the objective function Q to its minimal target value which is 0.47E-8. In this case, the union aggregation and the identification aggregation approaches give similar results. This is not so for other data bases.

To put the algorithm to a more severe test, the calculation is repeated with a zero matrix as the initial guess. The result upon convergence is again the same as that obtained the first time. Similar findings are obtained when the initial guess consists of all 1's. These results show the effectiveness of the identification algorithm and that the final answer is fairly independent of the initial guess.

To show the accuracy of the algorithm, we take, in turn, the resultant relation R computed by the union and identification methods, and see if using the original E's given by Eq. 94 will lead to the original C's given by Eq. 95. When Table 3 is used, the C's computed by E ∘ R are

$$c_1' = \frac{0.1}{y} + \frac{0.33}{1.1y} + \frac{0.55}{1.2y} + \frac{0.78}{1.3y} + \frac{1.}{1.4y}$$

$$c_2' = \frac{0.2}{y} + \frac{0.2}{1.1y} + \frac{1.}{1.2y} + \frac{0.2}{1.3y} + \frac{0.2}{1.4y} \qquad (96)$$

$$c_3' = \frac{1.}{y} + \frac{0.78}{1.1y} + \frac{0.55}{1.2y} + \frac{0.33}{1.3y} + \frac{0.1}{1.4y}$$

Note that they are different from the original information in Eq. 95. Hence, the union method of aggregation does not preserve the original inferences. When Table 4 is used instead, the original set of C's is recovered, showing that the original inferences are preserved by the identification method of aggregation.

VII. CONCLUSIONS AND RECOMMENDATIONS

TREATMENT OF UNCERTAINTIES

Many types of uncertainties coming from many different
sources are encountered in S/V assessment. Current assessment
methodologies consider all uncertainties to be random. This prac-
tice is undesirable because it forces nonrandom uncertainties to
be random, or ignores them altogether. Dire consequences may
result in either case.

It is our belief that improvement to S/V assessment can
be achieved simply by recognizing that there are at least two
major groups of uncertainties: random and fuzzy. Random un-
certainties are uncertainties which can be adequately modeled as
random parameters, functions or processes. Fuzzy uncertainties
include nonrandom uncertainties, as well as uncertainties re-
sulting from incomplete and imprecise information, subjective
judgment, ambiguity and vagueness. Much of the discussion in the
previous sections of this report is included to clarify and
support this belief. Examples of random and fuzzy uncertainties
are given. Their role and importance are delineated. Modeling
and analysis of the two groups of uncertainties and ways to
integrate them in an overall S/V assessment framework are
summarized.

MODELING AND ANALYSIS OF UNCERTAINTIES

Of the several extensions to current S/V assessment capa-
bilities studied using this approach, two are described in
more detail in this report: the use of random and stochastic
equations to model and analyze random uncertainties; and the use
of fuzzy sets and fuzzy logic to model and analyze fuzzy
uncertainties. The random equation study emphasized numerical
solution techniques and, in particular, the difference methods.
The fuzzy sets study emphasizes the treatment of expert experi-
ence and opinion and how it can be quantified in an S/V assess-
ment procedure. Major conclusions from these two studies and
recommendations for future efforts are described in the following
paragraphs.

Random/stochastic methods--This study indicates that the direct finite difference method is applicable to stochastic and random differential equations, and numerical techniques represent an attractive solution option to complement existing approximate and analytical methods. Immediate application to S/V assessment is feasible, but computational cost may impose a limit on the size of the problem which can be considered.

The limitation is due to the fact that a large sample size (thousands of samples) is needed to compute the sample statistics from the sample paths obtained by the difference methods. The large sample size is necessary, mainly because of errors in the simulation of the random processes. By comparison, the error due to the difference approximation itself is negligible. Hence, although the computational cost restiction is undesirable, it is not a limitation of the difference approach, but rather reflects the ineffectiveness of current computer algorithms for noise (pseudo-random number) generation.

Consequently, two research directions should be pursued to further extend the random/stochastic equation approach in S/V assessment. The first, which is to improve on random number generation techniques, belongs to the discipline of computer science. The second is to develop equations which govern the propagation of moments and numerical solutions of the moment equations.

For systems which can be modeled by stochastic (Ito) equations, there is the well-known Fokker-Planck equation which governs the transition probability density function and can be used to generate equations governing the moments of the response. However, no numerical works on these equations have been reported. For systems which can be modeled by general random equations, no general moment equations analogous to the Fokker-Planck exist. Many approximate methods have been attempted, but they all have limitations of one kind or another (see Ref. 22 and the recent work by Bennett, Ref. 47). Random equations are an extremely difficult group of mathematical problems, and much more research needs to be done before they can become application tools in S/V assessment.

Fuzzy models and methods--The study on fuzzy uncertainties focuses on two aspects of S/V assessment--namely, uncertainties

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associated with analysis models and the assessment of damage to structures. The selection is influenced largely by the author's research interest and does not imply a limitation on the applicability of the fuzzy set approach. The two pilot studies indicate the feasibility of the approach, as well as its versatility in modeling, in general, nonrandom uncertainties which have remained elusive. If one doubts the prevalence of such uncertainties, one needs only to select at random a page from any report related to S/V assessment and count the number of times linguistic terms such as "good," "severe," "satisfactory," etc., appear.

Perhaps several points need reiterating. Current practice treats uncertainties as if both the random and nonrandom, and the objective and subjective elements of the problem, have similar properties. This is incorrect and the distinction should be made. Fuzzy methods can be used to incorporate fuzzy, linguistic and judgmental data into the existing framework, and they complement existing random methods in this manner. Fuzzy models are not statistical ones in disguise, and they are not proposed to supplant random models used to model random uncertainties.

Future work should include more detailed studies of the two pilot studies initiated herein. A case study on damage assessment is in progress and results will be described in a separate report. The work described in this report centers on one basic tool in fuzzy set theory--namely, the fuzzy relation. Many other tools are available and should be explored. For applications in S/V assessment, the most promising appears to be fuzzy classification. Fuzzy classification and clustering techniques can be used to strengthen much of the work in evaluating and analyzing test data, which are hampered by measurement uncertainties, noise, scarcity of data, and subjective interpretation. Fuzzy reasoning procedures are useful tools to synthesize uncertainties (subjective as well as objective) in the S/V data streams within the framework of a knowledge-based assessment system.

CLOSING REMARKS

There are four issues which should always be raised in any discussion on modeling and analysis. They are:

(1) How to account for lack of understanding of
    some basic phenomenon;

(2) How to maximize use of sparse experimental
    data, and available engineering experience
    and expertise;

(3) How to assess the validity of the assumptions
    of the analysis; and

(4) How to assess the meaning of the results of
    the analysis.

For S/V assessment of protective structures, these issues are
especially relevant because of (1) the complexity of the phenom-
ena, (2) extremely sparse and indirect data, (3) inoperative
safety factors, and (4) dire consequences of miscalculation.
This study has provided a partial answer to these questions, but
is only a preliminary step in that direction.

The division of the uncertainties into only two main groups,
random and fuzzy, may be simplistic since there are other groups
of uncertainties. However, by acknowledging that there are
uncertainties other than the random variety and by seeking appro-
priate models for the two groups, the study constitutes a new
direction in S/V assessment. More research is obviously still
needed to make significant improvement to current technology.

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